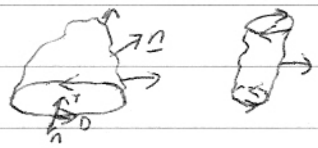
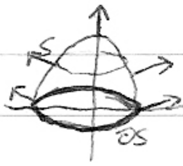
Stokes' Theorem

$\iint_S (\nabla \times \underline{F}) \cdot d\underline{S} = \oint_{\partial S} \underline{F} \cdot d\underline{r}$ $\quad d\underline{S} = \hat{\underline{n}}\, dA$



Example: Let S be the paraboloid $z = 9 - x^2 - y^2$, $z \geq 0$

Verify Stokes' Theorem for the vector field



$\underline{F} = (2z - y)\underline{i} + (x + z)\underline{j} + (3x - 2y)\underline{k}$

$$\nabla \times \underline{F} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2z-y & x+z & 3x-2y \end{vmatrix} = (-2-1)\underline{i} + (2-3)\underline{j} + (1+1)\underline{k}$$
$$= -3\underline{i} - \underline{j} + 2\underline{k}$$

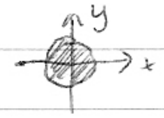
S is the graph of $z = f(x,y)$ so we parameterise S as

$\underline{r}(x,y) = x\underline{i} + y\underline{j} + (9 - x^2 - y^2)\underline{k}$ $\quad x^2 + y^2 \leq 9$

$$\underline{n} = \frac{\partial \underline{r}}{\partial x} \times \frac{\partial \underline{r}}{\partial y} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 1 & 0 & -2x \\ 0 & 1 & -2y \end{vmatrix} = 2x\underline{i} + 2y\underline{j} + \underline{k} = \underline{n} \quad \text{(points upward)}$$

$$\iint_S (\nabla \times \underline{F}) \cdot d\underline{S} = \iint_{x^2+y^2 \leq 9} (-3\underline{i} - \underline{j} + 2\underline{k}) \cdot (2x\underline{i} + 2y\underline{j} + \underline{k})\, dx\, dy \qquad d\underline{S} = \underline{n}\, dx\, dy.$$

$$= \iint_{x^2+y^2 \leq 9} (-6x - 2y + 2)\, dx\, dy = \iint_{x^2+y^2 \leq 9} 2\, dx\, dy \quad \text{(by symmetry)}$$



$= 2 \times \text{Area} = 2 \times 9\pi = 18\pi$

On the other hand, to calculate $\oint_{\partial S} \underline{F} \cdot d\underline{r}$ we parameterise $\partial S = C$ as

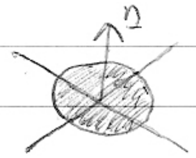
$\underline{r}(t) = 3\cos t\, \underline{i} + 3\sin t\, \underline{j} + 0\underline{k}$ $\quad 0 \leq t \leq 2\pi$.

$\underline{F}(\underline{r}(t)) = -3\sin t\, \underline{i} + 3\cos t\, \underline{j} + (9\cos t - 6\sin t)\underline{k}$

$\frac{d\underline{r}}{dt} = -3\sin t\, \underline{i} + 3\cos t\, \underline{j} + 0\underline{k}$ $\quad \oint \underline{F} \cdot d\underline{r} = \int_{t=0}^{2\pi} \underline{F}(\underline{r}(t)) \cdot \frac{d\underline{r}(t)}{dt}\, dt.$

$$= \int_{t=0}^{2\pi} (9\sin^2 t + 9\cos^2 t)\, dt = \int_{t=0}^{2\pi} 9\, dt = 18\pi$$

Example 2: Let $\tilde{S}$ be the disc $x^2 + y^2 \leq 9$, $z = 0$.



Calculate $\iint_{\tilde{S}} (\nabla \times \underline{F}) \cdot d\underline{S}$ where $\underline{F}$ is as before.

(and $\underline{n}$ to be the upward normal)

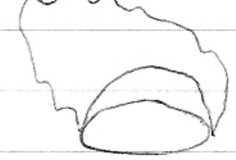
$\partial \tilde{S} = C = \partial S$, so by Stokes' Theorem

$$\iint_{\tilde{S}} (\nabla \times \underline{F}) \cdot d\underline{S} = \oint_{\partial \tilde{S}} \underline{F} \cdot d\underline{r} = \oint_{\partial S} \underline{F} \cdot d\underline{r} = \iint_S (\nabla \times \underline{F}) \cdot d\underline{S} = 18\pi.$$

Corollary of Stokes' Theorem

Let S and $\tilde{S}$ be two (orientable) smooth surfaces with the same boundary.

Then $\iint_S (\nabla \times \underline{F}) \cdot d\underline{S} = \iint_{\tilde{S}} (\nabla \times \underline{F}) \cdot d\underline{S}$



Look at this result again when we consider the divergence theorem.

Conservative Vector Fields.

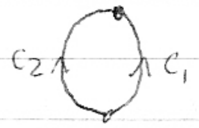
Definition: A continuous vector field $\underline{F}$ has path-independent line integrals (in the region U) if

$$\int_{C_1} \underline{F} \cdot d\underline{r} = \int_{C_2} \underline{F} \cdot d\underline{r}$$

for any two piecewise C^1 oriented curves C_1 and C_2 which start at the same point and end at the same point.

Proposition: Let $\underline{F}$ be a continuous vector field. Then $\underline{F}$ has path-independent integrals if and only if $\oint_C \underline{F} \cdot d\underline{r} = 0$.

Proof.



$$\oint_C \underline{F} \cdot d\underline{r} = \int_{C_1} \underline{F} \cdot d\underline{r} - \int_{C_2} \underline{F} \cdot d\underline{r}$$