Flux Integrals

Calculating flux integrals: $\iint_S \underline{F}\cdot\hat{\underline{n}}\,dA = \iint_S \underline{F}\cdot d\underline{s}$ where $d\underline{s} = \hat{\underline{n}}\,dA$.

$dA = \left|\frac{\partial \underline{r}}{\partial s}\times\frac{\partial \underline{r}}{\partial t}\right| ds\,dt$ Let $\underline{n} = \frac{\partial \underline{r}}{\partial s}\times\frac{\partial \underline{r}}{\partial t}$ Then $d\underline{s} = \left(\frac{\partial \underline{r}}{\partial s}\times\frac{\partial \underline{r}}{\partial t}\right) ds\,dt$.

Example: Let S be the parameterised surface given by

$\underline{r}(s,t) = s\cos t\,\underline{i} + s\sin t\,\underline{j} + t\,\underline{k}$ $0 \le s \le 1$ $0 \le t \le 2\pi$

twisted ribbon-helicoid and let $\underline{F} = x\underline{i} + y\underline{j} + (z - 2y)\underline{k}$

evaluate the flux integral $\iint_S \underline{F}\cdot d\underline{s}$

$$\underline{n} = \frac{\partial \underline{r}}{\partial s}\times\frac{\partial \underline{r}}{\partial t} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ \cos t & \sin t & 0 \\ -s\sin t & s\cos t & 1 \end{vmatrix} = \sin t\,\underline{i} - \cos t\,\underline{j} + (s\cos^2 t + s\sin^2 t)\underline{k}$$

$\underline{n} = \sin t\,\underline{i} - \cos t\,\underline{j} + s\underline{k}$.

$\underline{F}(\underline{r}(s,t)) = s\cos t\,\underline{i} + s\sin t\,\underline{j} + (t - 2s\sin t)\underline{k}$ $\iint_S \underline{F}\cdot d\underline{s} = \iint_S (\underline{F}\cdot\underline{n})\,ds\,dt$.

$\underline{F}\cdot\underline{n} = s\sin t\cos t - s\cos t\sin t + st - 2s^2\sin t = st - 2s^2\sin t$.

$$\iint_S \underline{F}\cdot d\underline{s} = \int_{s=0}^{1}\int_{t=0}^{2\pi} (\underline{F}\cdot\underline{n})\,dt\,ds = \int_{s=0}^{1}\int_{t=0}^{2\pi} (st - 2s^2\sin t)\,dt\,ds.$$

$$= \int_{s=0}^{1}\left[\tfrac{1}{2}st^2 + 2s^2\cos t\right]_{t=0}^{2\pi} ds = \int_{s=0}^{1} 2s\pi^2\,ds$$

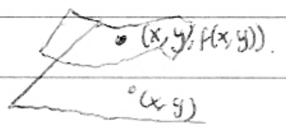
$$= \left[s^2\pi^2\right]_0^1 = \pi^2$$

Flux integrals when the surface is the graph of a function

$z = f(x,y)$. We use x and y as the parameters, then the surface is

$\underline{r}(x,y) = x\underline{i} + y\underline{j} + f(x,y)\underline{k}$

$$\underline{n} = \frac{\partial \underline{r}}{\partial x}\times\frac{\partial \underline{r}}{\partial y} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 1 & 0 & f_x \\ 0 & 1 & f_y \end{vmatrix} \qquad f_x = \frac{\partial f}{\partial x}$$



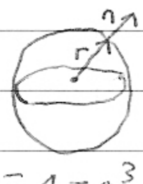
$\underline{n} = -f_x\underline{i} - f_y\underline{j} + \underline{k}$

$$\iint_S \underline{F}\cdot d\underline{s} = \iint \underline{F}(x,y,f(x,y))\cdot(-f_x\underline{i} - f_y\underline{j} + \underline{k})\,dx\,dy.$$

In some examples it is worth thinking about the geometry before calculating.

Example: Let $\underline{F} = x\underline{i} + y\underline{j} + z\underline{k}$ and S the surface of the sphere $x^2+y^2+z^2 = a^2$

Calculate $\iint_S \underline{F}\cdot d\underline{s}$



$\underline{F} = \underline{r}$ $d\underline{s} = \hat{\underline{n}}\,dA = \hat{\underline{r}}\,dA$

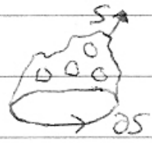
$\underline{F}\cdot d\underline{s} = \underline{r}\cdot\hat{\underline{r}}\,dA = r(\hat{\underline{r}}\cdot\hat{\underline{r}})\,dA = r\,dA$

So $\iint_S \underline{F}\cdot d\underline{s} = \iint_S r\,dA = a\iint_S dA = a \times \text{area of sphere} = a \times 4\pi a^2 = 4\pi a^3$

III INTEGRAL THEOREMS

Relationship between line integrals, flux integrals, and volume integrals, using grad, div, and curl.

Theorem (Stokes' Theorem): Let S be a bounded, piecewise smooth, orientable surface in $\mathbb{R}^3$, with boundary ∂S that consists of a finite number of piecewise C^1 simple closed curves.



Let $\underline{F}$ be a C^1 vector field whose domain includes S

Then $$\iint_S (\underline{\nabla} \times \underline{F}) \cdot d\underline{S} = \oint_{\partial S} \underline{F} \cdot d\underline{r}$$

$\oint$ - line integral where you finish where start