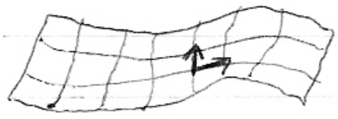
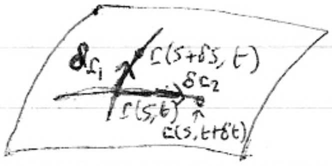
$(s,t) \mapsto \underline{r}(s,t)$ $\quad s_0 \le s \le s_1, \quad t_0 \le t \le t_1$

$\frac{\partial \underline{r}}{\partial s}$ $\frac{\partial \underline{r}}{\partial t}$ tangent to the surface $\Delta = \frac{\partial \underline{r}}{\partial s} \times \frac{\partial \underline{r}}{\partial t}$.



Area of a surface



Suppose we start at $\underline{r}(s,t)$ and move to $\underline{r}(s+\delta s, t)$. Let $\delta \underline{r}_1 = \underline{r}(s+\delta s, t) - \underline{r}(s,t)$.

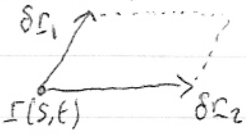
By Taylors Theorem $\underline{r}(s+\delta s, t) = \underline{r}(s,t) + \delta s \frac{\partial \underline{r}}{\partial s}(s,t) + O(\delta s^2)$.

($f(x+h) = f(x) + \frac{df}{dx}(x) \cdot h + O(h^2)$ where O is the order symbol)

Similarly if we start at $\underline{r}(s,t)$ and move to $\underline{r}(s, t+\delta t)$.

Let $\delta \underline{r}_2 = \underline{r}(s, t+\delta t) - \underline{r}(s,t)$ $\quad \underline{r}(s,t+\delta t) = \underline{r}(s,t) + \delta t \frac{\partial \underline{r}}{\partial t}(s,t) + O(\delta t^2)$.

So $\delta \underline{r}_1 \simeq \delta s \frac{\partial \underline{r}}{\partial s}(s,t)$ and $\delta \underline{r}_2 = \delta t \frac{\partial \underline{r}}{\partial t}(s,t)$.



$\delta A = |\delta \underline{r}_1 \times \delta \underline{r}_2| \simeq \left|\frac{\partial \underline{r}}{\partial s} \times \frac{\partial \underline{r}}{\partial t}\right| \delta s \, \delta t$.

In the limit we get $dA = \left|\frac{\partial \underline{r}}{\partial s} \times \frac{\partial \underline{r}}{\partial t}\right| ds\,dt$

So the area of the curved surface is

$$A = \int_{t=t_0}^{t_1} \int_{s=s_0}^{s_1} dA = \int_{t=t_0}^{t_1} \int_{s=s_0}^{s_1} \left|\frac{\partial \underline{r}}{\partial s}(s,t) \times \frac{\partial \underline{r}}{\partial t}(s,t)\right| ds\,dt.$$

Surface area of a sphere.

$\underline{r}(s,t) = a\cos s \sin t \, \underline{i} + a \sin s \sin t \, \underline{j} + a \cos t \, \underline{k}$ $\quad 0 \le t \le \pi \quad 0 \le s \le 2\pi$.

$\frac{\partial \underline{r}}{\partial s} = -a\sin s \sin t \, \underline{i} + a\cos s \sin t \, \underline{j} + 0\underline{k}$ $\quad \frac{\partial \underline{r}}{\partial t} = a\cos s \cos t \, \underline{i} + a \sin s \cos t \, \underline{j} - a \sin t \, \underline{k}$

$$\frac{\partial \underline{r}}{\partial s} \times \frac{\partial \underline{r}}{\partial t} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ -a\sin s \sin t & a\cos s \sin t & 0 \\ a\cos s \cos t & a \sin s \cos t & -a\sin t \end{vmatrix}$$

$\frac{\partial \underline{r}}{\partial s} \times \frac{\partial \underline{r}}{\partial t} = -a^2 \sin t (\cos s \sin t \, \underline{i} + \sin s \sin t \, \underline{j} + \cos t \, \underline{k})$

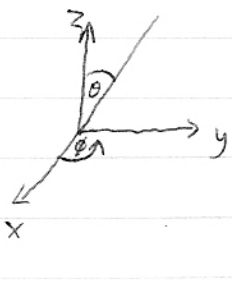
$\left|\frac{\partial \underline{r}}{\partial s} \times \frac{\partial \underline{r}}{\partial t}\right| = a^2 \sin t (\cos^2 s \sin^2 t + \sin^2 s \sin^2 t + \cos^2 t)^{1/2}$

$= a^2 \sin t ((\cos^2 s + \sin^2 s)(\sin^2 t + \cos^2 t))^{1/2} = a^2 \sin t$

for a sphere $dA = a^2 \sin t \, ds\,dt$.

Note: a is just the radius of the sphere

s and t are the usual polar co-ordinates ϕ and θ



So $dA = r^2 \sin\theta \, d\theta \, d\phi$

Area of a Sphere

$$A=\int_{\theta=0}^{\pi}\int_{\phi=0}^{2\pi} r^2\sin\theta\,d\phi\,d\theta = 2\pi r^2\int_0^{\pi}\sin\theta\,d\theta = 2\pi r^2\left[-\cos\theta\right]_0^{\pi} = 4\pi r^2.$$

Cylinder $\underline{r}(s,t) = a\cos s\,\underline{i} + a\sin s\,\underline{j} + t\,\underline{k}$ $\quad 0\le s\le 2\pi \quad 0\le t\le h.$

$$\frac{\partial \underline{r}}{\partial s}\times\frac{\partial \underline{r}}{\partial t} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ -a\sin s & a\cos s & 0 \\ 0 & 0 & 1 \end{vmatrix} = a\cos s\,\underline{i} + a\sin s\,\underline{j}$$

$$\left|\frac{\partial \underline{r}}{\partial s}\times\frac{\partial \underline{r}}{\partial t}\right| = a\left(\cos^2 s+\sin^2 s\right)^{1/2} = a.$$

$$dA = \left|\frac{\partial \underline{r}}{\partial s}\times\frac{\partial \underline{r}}{\partial t}\right| ds\,dt = a\,ds\,dt.$$

a is the radius r, s is θ, t is z in cylindrical polar co-ordinates.

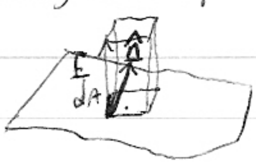
So we get $dA = r\,d\theta\,dz$ in cylindrical polar co-ordinates.

$$\text{Area} = \int_{\theta=0}^{2\pi}\int_{z=0}^{h} r\,d\theta\,dz = r\,2\pi h = 2\pi r h.$$

Flux Integrals.

A flux integral measures the 'flow' of a vector field through a surface S.

$$\iint_S \underline{F}\cdot\hat{\underline{n}}\,dA$$



Volume = base × height $= dA\cdot(\underline{F}\cdot\hat{\underline{n}})$.

If we think of $\underline{F}$ as the velocity of the fluid

$\iint_S \underline{F}\cdot\hat{\underline{n}}\,dA$ measures the rate of flow of the fluid through the surface.