Friday 24TH October 2008 MATH2045 - Vector Calculus + Complex Variable Theory.

Line Integrals

Proposition: Let γ_{AB} be a curve from A (with position vector $\underline{r}_A$) to B (with position vector $\underline{r}_B$) Then $\int_{\gamma_{AB}} \nabla \phi \cdot d\underline{r} = \phi(\underline{r}_B) - \phi(\underline{r}_A)$.

Proof: let γ_{AB} be parametrised by $\underline{r}(t)$ $t_0 \leq t \leq t_1$

then $\int_{\gamma_{AB}} \nabla\phi \cdot d\underline{r} = \int_{t=t_0}^{t_1} (\nabla\phi(\underline{r}(t)) \cdot \frac{d\underline{r}}{dt})\, dt$.

Now $\nabla \phi = \frac{\partial\phi}{\partial x}\underline{i} + \frac{\partial\phi}{\partial y}\underline{j} + \frac{\partial\phi}{\partial z}\underline{k}$ Let $\underline{r}(t) = x(t)\underline{i} + y(t)\underline{j} + z(t)\underline{k}$

Then $\frac{d\underline{r}}{dt} = \frac{dx}{dt}\underline{i} + \frac{dy}{dt}\underline{j} + \frac{dz}{dt}\underline{k}$ So $\nabla\phi \cdot \frac{d\underline{r}}{dt} = \frac{\partial\phi}{\partial x}\frac{dx}{dt} + \frac{\partial\phi}{\partial y}\frac{dy}{dt} + \frac{\partial\phi}{\partial z}\frac{dz}{dt}$

$= \frac{d\phi}{dt}$ (by the chain rule)

So $\int_{\gamma_{AB}} \nabla\phi \cdot d\underline{r} = \int_{t=t_0}^{t_1} \frac{d\phi}{dt}\, dt = [\phi(\underline{r}(t))]_{t=t_0}^{t_1} = \phi(\underline{r}(t_1)) - \phi(\underline{r}(t_0))$.

$= \phi(\underline{r}_B) - \phi(\underline{r}_A)$.

SURFACE INTEGRALS

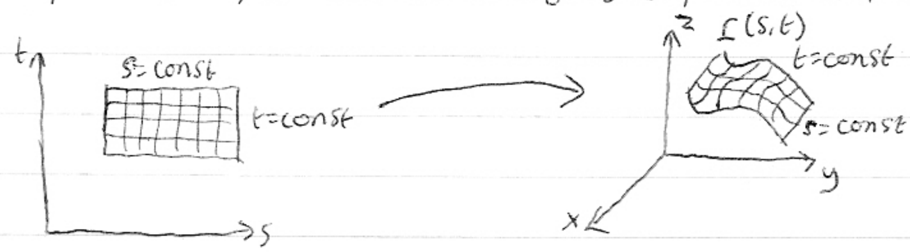
Surfaces i) graph of a function of two variables $z = f(x,y)$ e.g. $z = x^2 + 4y^2$

ii) level surface $F(x,y,z) = c$ e.g. $x^2 + y^2 + z^2 = c^2$

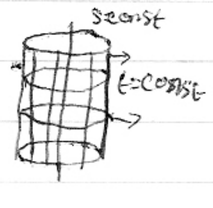
iii) as a parameterised surface

$(s,t) \mapsto (x(s,t), y(s,t), z(s,t))$ $s_0 \leq s \leq s_1$ $t_0 \leq t \leq t_1$

We will usually think of x, y, z as the component of a vector and write the surface as $(s,t) \mapsto \underline{r}(s,t)$ $s_0 \leq s \leq s_1$ $t_0 \leq t \leq t_1$



(s,t) are coordinates on the surface.

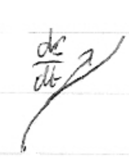


Example $\underline{r}(s,t) = a\cos s\,\underline{i} + a\sin s\,\underline{j} + t\underline{k}$

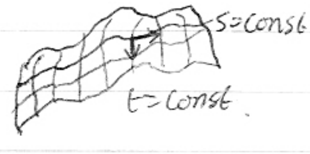
$-\pi \leq s \leq \pi$ $-1 \leq t \leq 1$

$\underline{r}(s,t) = a\cos s \sin t\,\underline{i} + a \sin s \sin t + a\cos t\,k$

$0 \leq s \leq 2\pi$ $0 \leq t \leq \pi$



For a curve $\underline{r}(t)$ $\frac{dr}{dt}$ is tangent to it.



For a surface $\underline{r}(s,t)$ $\frac{\partial r}{\partial s}$ is tangent to the curve $t = \text{const}$

and $\frac{\partial r}{\partial t}$ is tangent to the curve $s = \text{const}$

both of these are tangent to the surface.

A normal to the surface is given by $\underline{n} = \frac{\partial \underline{r}}{\partial s} \times \frac{\partial \underline{r}}{\partial t}$

Cylinder example: $\frac{\partial \underline{r}}{\partial s} = -a\sin s\,\underline{i} + a\cos s\,\underline{j} + 0\underline{k}$ $\quad \frac{\partial \underline{r}}{\partial t} = 0\underline{i} + 0\underline{j} + 1\underline{k}$

$$\underline{n} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ -a\sin s & a\cos s & 0 \\ 0 & 0 & 1 \end{vmatrix} = a\cos s\,\underline{i} + a\sin s\,\underline{j}$$

Sphere example: $\frac{\partial \underline{r}}{\partial s} = -a\sin s\sin t\,\underline{i} + a\cos s\sin t\,\underline{j} + 0\underline{k}$

$\frac{\partial \underline{r}}{\partial t} = a\cos s\cos t\,\underline{i} + a\sin s\cos t\,\underline{j} - a\sin t\,\underline{k}$

$$\underline{n} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ -a\sin s\sin t & a\cos s\sin t & 0 \\ a\cos s\cos t & a\sin s\cos t & -a\sin t \end{vmatrix} = -a^2\sin t(\cos s\sin t\,\underline{i} + \sin s\sin t\,\underline{j} + \cos t\,\underline{k})$$

$$= -a^2\sin t\,\hat{\underline{r}}$$

so $\hat{n} = \hat{r}$