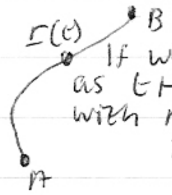
Thursday 23rd October 2008 MATH2045 - Vector Calculus + Complex Variable Theory

Line Integrals

$\int_\gamma \underline{F}\cdot d\underline{r}$



If we write γ as $t \mapsto \underline{r}(t)$ with $A=\gamma(t_0)$, $B=\gamma(t_1)$

Then $\int_\gamma \underline{F}\cdot d\underline{r} = \int_{t=t_0}^{t} \left(\underline{F}(\underline{r}(t))\cdot \frac{d\underline{r}}{dt}\right) dt$

one can show that the RHS does not depend upon how we parametize γ

Example 1 Given $\underline{F} = 2xy\,\underline{i} + (x^2 - z^2)\underline{j} - 3xz^2\underline{k}$

evaluate $\int_\gamma \underline{F}\cdot d\underline{r}$ where γ is

a) the curve $\underline{r}(t) = \underbrace{2t}_{x}\underline{i} + \underbrace{t^3}_{y}\underline{j} + \underbrace{3t^2}_{z}\underline{k}$ $\quad 0 \le t \le 1$

from O to $A=(2,1,3)$.

$\underline{F}(\underline{r}(t)) = 2\cdot 2t\cdot t^3\,\underline{i} + (4t^2 - 9t^4)\underline{j} - 3\cdot 2t\cdot 9t^4\,\underline{k}$

$= 4t^4\underline{i} + (4t^2 - 9t^4)\underline{j} - 54t^5\underline{k}$

$\frac{d\underline{r}}{dt} = 2\underline{i} + 3t^2\underline{j} + 6t\,\underline{k}$

$\int_\gamma \underline{F}\cdot d\underline{r} = \int_{t=0}^{1} \left(\underline{F}\cdot \frac{d\underline{r}}{dt}\right)dt = \int_{t=0}^{1} (8t^4 + 12t^4 - 27t^6 - 324t^6)\,dt$

$= \left[4t^5 - \frac{351}{7}t^7\right]_0^1 = -\frac{323}{7}$

b) keep the same $\underline{F}$ but evaluate the integral along the straight line from O to A.

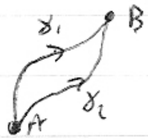
$\underline{r}(t) = \underbrace{2t}_{x}\underline{i} + \underbrace{t}_{y}\underline{j} + \underbrace{3t}_{z}\underline{k}$ $\quad 0 \le t \le 1$

$\underline{F}(\underline{r}(t)) = 2\cdot 2t\cdot t\,\underline{i} + (4t^2 - 9t^2)\underline{j} - 3\cdot 2t\cdot 9t^2\,\underline{k}$

$= 4t^2\underline{i} - 5t^2\underline{j} - 54t^3\underline{k}$ $\qquad \frac{d\underline{r}}{dt} = 2\underline{i} + \underline{j} + 3\underline{k}$

$\int_\gamma \underline{F}\cdot d\underline{r} = \int_{t=0}^{1} \left(\underline{F}\cdot\frac{d\underline{r}}{dt}\right)dt = \int_{t=0}^{1} (8t^2 - 5t^2 - 162t^3)\,dt = \left[t^3 - \frac{81}{2}t^4\right]_0^1 = -\frac{79}{2}$

different from answer a).



So we see from this example that $\int_\gamma \underline{F}\cdot d\underline{r}$

this depends (in general) on the path γ, not just on the end points. $\int_{\gamma_1} \underline{F}\cdot d\underline{r} \ne \int_{\gamma_2} \underline{F}\cdot d\underline{r}$.

Example 2. Let $\phi = xy^2z$ and $\underline{F} = \nabla\phi$ calculate $\int_\gamma \underline{F}\cdot d\underline{r}$ for the two curves in the previous example.

$\underline{F} = y^2z\,\underline{i} + 2xyz\,\underline{j} + xy^2\underline{k}$

a) $\underline{r}(t) = 2t\underline{i} + t^3\underline{j} + 3t^2\underline{k}$ $\qquad \underline{F}(\underline{r}(t)) = 3t^8\underline{i} + 12t^6\underline{j} + 2t^7\underline{k}$

$\frac{d\underline{r}}{dt} = 2\underline{i} + 3t^2\underline{j} + 6t\,\underline{k}$

$\int_\gamma \underline{F}\cdot d\underline{r} = \int_{t=0}^{1} \left(\underline{F}\cdot\frac{d\underline{r}}{dt}\right)dt = \int_{t=0}^{1} 6t^8 + 36t^8 + 12t^8 = \left[\frac{54}{9}t^9\right]_0^1 = 6.$

b) $\underline{r}(t) = 2t\underline{i} + t\underline{j} + 3t\underline{k}$ $\quad 0 \le t \le 1$ $\qquad \underline{F}(\underline{r}(t)) = 3t^3\underline{i} + 12t^3\underline{j} + 2t^3\underline{k}$

$\frac{d\underline{r}}{dt} = 2\underline{i} + \underline{j} + 3\underline{k}$

$\int_\gamma \underline{F}\cdot d\underline{r} = \int_{t=0}^{1} \left(\underline{F}\cdot\frac{d\underline{r}}{dt}\right)dt = \int_{t=0}^{1} (6t^3 + 12t^3 + 6t^3)\,dt = \left[\frac{24}{4}t^4\right]_0^1 = 6.$

Look at $\phi(0,0,0)=0$ $\quad \phi(2,1,3)=2\cdot1^2\cdot3=6$.

In this example we found $\int_\gamma \underline{\nabla}\phi\cdot d\underline{r} = \phi(A)-\phi(0)$.

This is true in general.

If $\underline{F}=\underline{\nabla}\phi$ then $\int_\gamma \underline{F}\cdot d\underline{r} = \phi(B)-\phi(A)$ where γ is any curve between A and B.

So for a general vector field $\int_\gamma \underline{F}\cdot d\underline{r}$ depends on γ.

for a vector field $\underline{F}=\underline{\nabla}\phi$ then $\int_\gamma \underline{F}\cdot d\underline{r}$ only depends on the end points.

Why might we be interested in calculating line integrals?

Example: Calculate the work done by the vector field $\underline{F}=-y\underline{i}+x\underline{j}$ moving along the curve $\underline{r}(t)=\cos(\pi t)\underline{i}+\sin(\pi t)\underline{j}$ $\quad 0\le t\le 1$

Work done by $\underline{F}$ is force $\times$ distance moved in the direction of the force.

$$\lim_{n\to\infty}\sum_{i=1}^{n} \underline{F}_i\,\delta\underline{r}_i = \int_\gamma \underline{F}\cdot d\underline{r}$$

$\underline{F}(\underline{r}(t)) = -\sin(\pi t)\underline{i}+\cos(\pi t)\underline{j}$ $\qquad \frac{d\underline{r}}{dt} = -\pi\sin(\pi t)\underline{i}+\pi\cos(\pi t)\underline{j}$

$$\int_\gamma \underline{F}\cdot d\underline{r} = \int_{t=0}^{1} (\pi\sin^2(\pi t)+\pi\cos^2(\pi t))\,dt = \int_0^1 \pi\,dt = \pi$$