Wednesday 22nd October 2008 MATH2045 - Vector Calculus + Complex Variable Theory.

Combining operations

Theorem: Let $\underline{F}$ be a C^2 vector field then $\text{div}(\text{curl}\,\underline{F}) = 0$ $\quad \underline{\nabla}\cdot(\underline{\nabla}\times\underline{F}) = 0.$

Proof: $\underline{F} = F_1\underline{i} + F_2\underline{j} + F_3\underline{k}$

$$\underline{G} = \underline{\nabla}\times\underline{F} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix}$$

$$= \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right)\underline{i} + \left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\right)\underline{j} + \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial z}\right)\underline{k}$$

$$\underline{\nabla}\cdot\underline{G} = \frac{\partial G_1}{\partial x} + \frac{\partial G_2}{\partial y} + \frac{\partial G_3}{\partial z} = \frac{\partial^2 F_3}{\partial x\partial y} - \frac{\partial^2 F_2}{\partial x\partial z} + \frac{\partial^2 F_1}{\partial y\partial z} - \frac{\partial^2 F_3}{\partial y\partial x} + \frac{\partial^2 F_2}{\partial z\partial x} - \frac{\partial^2 F_1}{\partial z\partial y}$$

$$= 0.$$

$\text{curl}(\text{grad}\,\phi) = 0$

$\text{div}(\text{curl}\,\underline{F}) = 0$

$d^2 = 0$

Scalar field

$\downarrow$ GRAD d

Vector field

$\downarrow$ CURL d

Vector field

$\downarrow$ DIV d.

Scalar field

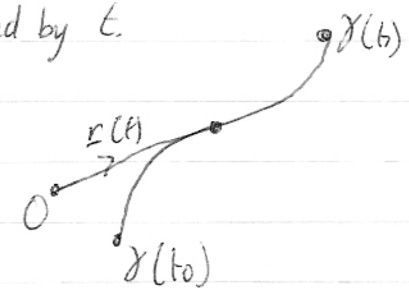
Vector Identities

1) $\underline{\nabla}(\phi\psi) = \psi\underline{\nabla}\phi + \phi\underline{\nabla}\psi$
2) $\underline{\nabla}\cdot(\phi\underline{F}) = \underline{\nabla}\phi\cdot\underline{F} + \phi\underline{\nabla}\cdot\underline{F}$
3) $\underline{\nabla}\times(\phi\underline{F}) = \underline{\nabla}\phi\times\underline{F} + \phi(\underline{\nabla}\times\underline{F})$
4) $\underline{\nabla}\cdot(\underline{F}\times\underline{G}) = \underline{G}\cdot(\underline{\nabla}\times\underline{F}) - \underline{F}\cdot(\underline{\nabla}\times\underline{G}).$
5) $\underline{\nabla}\times(\underline{\nabla}\phi) = 0$
6) $\underline{\nabla}\cdot(\underline{\nabla}\times\underline{F}) = 0$
7) $\underline{\nabla}\cdot\underline{\nabla}\phi = \nabla^2\phi$
8) $\underline{\nabla}\times(\underline{\nabla}\times\underline{F}) = \underline{\nabla}(\underline{\nabla}\cdot\underline{F}) - \nabla^2\underline{F}.$

$$\nabla^2\underline{F} = \nabla^2F_1\underline{i} + \nabla^2F_2\underline{j} + \nabla^2F_3\underline{k}$$

II Integrals of Vector fields

Let $\underline{F}(\underline{r})$ be a vector field and $\gamma(t)$ a curve paramerised by t.

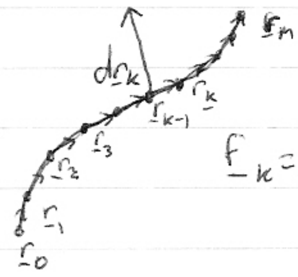


Let $\underline{r}(t)$ be the position vector of the point on the curve $\gamma(t)$.

I want to define $\int_\gamma \underline{F}\cdot d\underline{r}$ line integral of the vector field $\underline{F}$ along the curve γ.

In the usual way we can think of this as a limit

$$\int_\gamma \underline{F}\cdot d\underline{r} = \lim_{m\to\infty}\sum_{j=1}^{m}\underline{F}_j\cdot\delta\underline{r}_j.$$



$\underline{F}_k = \underline{F}(\underline{c}_k)$

To calculate the integral we first parameterise the curve γ

We write γ as $t \mapsto \underline{r}(t)$, $t_0 \leq t \leq t_1$

Then we write $d\underline{r}$ as $\frac{d\underline{r}}{dt}\, dt$

Then we look at $\underline{F}(\underline{r}(t))$ so $\underline{F}(\underline{r}) \cdot d\underline{r}$ becomes $\left(\underline{F}(\underline{r}(t)) \cdot \frac{d\underline{r}(t)}{dt}\right) dt$.

$$\int_{\gamma} \underline{F} \cdot d\underline{r} = \underbrace{\int_{t=t_0}^{t_1} \left(\underline{F}(\underline{r}(t)) \cdot \frac{d\underline{r}}{dt}\right) dt}_{} = \int_{t=t_0}^{t_1} f(t)\, dt$$

line integral   just some function $f(t)$ ordinary integral