Friday 17TH October 2008 MATH2045 - Vector Calculus + Complex Variable Theory.

Laplacian

Definition $\nabla^2\phi = \text{div}(\text{grad}\,\phi) = \underline{\nabla}\cdot(\underline{\nabla}\phi)$

In Cartesian co-ordinates: $\nabla^2\phi = \frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}+\frac{\partial^2\phi}{\partial z^2}$

Example $\phi(x,y,z)=e^x y^3\sin z$ $\quad \nabla\phi = \frac{\partial\phi}{\partial x}\underline{i}+\frac{\partial\phi}{\partial y}\underline{j}+\frac{\partial\phi}{\partial z}\underline{k}$

$$\underline{f}=\underline{\nabla}\phi = \underbrace{e^x y^3\sin z}_{f_1}\,\underline{i} + \underbrace{3e^x y^2\sin z}_{f_2}\,\underline{j} + \underbrace{e^x y^3\cos z}_{f_3}\,\underline{k}$$

$$\underline{\nabla}\cdot\underline{f} = \frac{\partial f_1}{\partial x}+\frac{\partial f_2}{\partial y}+\frac{\partial f_3}{\partial z} = e^x y^3\sin z + 6e^x y\sin z - e^x y^3\sin z$$

$$\nabla^2\phi = 6e^x y\sin z = \frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}+\frac{\partial^2\phi}{\partial z^2}$$

Proposition $\underline{\nabla}\cdot(\phi\underline{f}) = \underline{\nabla}\phi\cdot\underline{f}+\phi(\underline{\nabla}\cdot\underline{f})$

Proof: See problem sheet 2..

Proposition $\underline{\nabla}\cdot(\underline{f}\times\underline{G}) = ?$ This requires us to define a new differential operator $\underline{\nabla}\times\underline{f}$ called the curl

Definition: Let $\underline{f}$ be a differentiable vector field then the curl of $\underline{f}$ is defined by $\text{curl}(\underline{f}) = \underline{\nabla}\times\underline{f} = \begin{vmatrix}\underline{i} & \underline{j} & \underline{k}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ f_1 & f_2 & f_3\end{vmatrix}$

$$= \left(\frac{\partial f_3}{\partial y}-\frac{\partial f_2}{\partial z}\right)\underline{i} + \left(\frac{\partial f_1}{\partial z}-\frac{\partial f_3}{\partial x}\right)\underline{j} + \left(\frac{\partial f_2}{\partial x}-\frac{\partial f_1}{\partial y}\right)\underline{k}$$

$(x,y,z)\leftrightarrow(x_1,x_2,x_3)$ $\frac{\partial f}{\partial x_a} = f_{,a}$ so $= (f_{3,2}-f_{2,3})\underline{i} + (f_{1,3}-f_{3,1})\underline{j} + (f_{2,1}-f_{1,2})\underline{k}$

Example Let $\underline{f} = xy^2\underline{i}+e^z\underline{j}+y\sin x e^z\underline{k}$ Calculate $\underline{\nabla}\times\underline{f}$ $= \begin{vmatrix}\underline{i} & \underline{j} & \underline{k}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ xy^2 & e^z & y\sin x e^z\end{vmatrix}$

$$= (\sin x e^z - e^z)\underline{i} - y\cos x e^z\underline{j} - 2xy\underline{k}$$

Example: $\phi = x^2yz^3$ $\underline{\nabla}\phi = \frac{\partial\phi}{\partial x}\underline{i}+\frac{\partial\phi}{\partial y}\underline{j}+\frac{\partial\phi}{\partial z}\underline{k} = \underbrace{2xyz^3}_{f_1}\underline{i}+\underbrace{x^2z^3}_{f_2}\underline{j}+\underbrace{3x^2yz^2}_{f_3}\underline{k} = \underline{f}$

$$\underline{\nabla}\times\underline{f} = \begin{vmatrix}\underline{i} & \underline{j} & \underline{k}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ 2xyz^3 & x^2z^3 & 3x^2yz^2\end{vmatrix} = (3x^2z^2-3x^2z^2)\underline{i}+(6xyz^2-6xyz^2)\underline{j}+(2xz^3-2xz^3)\underline{k}$$
$$= \underline{0}$$

Theorem: Let ϕ be a C^2 ~~vector~~ scalar field then $\underline{\nabla}\times(\underline{\nabla}\phi) = \underline{0}$

C^2 = twice differentiable with continuous 2nd derivative

proof: $\nabla\phi = \frac{\partial\phi}{\partial x}\underline{i}+\frac{\partial\phi}{\partial y}\underline{j}+\frac{\partial\phi}{\partial z}\underline{k}$

$$\underline{\nabla}\times\underline{\nabla}\phi = \begin{vmatrix}\underline{i} & \underline{j} & \underline{k}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ \frac{\partial\phi}{\partial x} & \frac{\partial\phi}{\partial y} & \frac{\partial\phi}{\partial z}\end{vmatrix} = \left(\frac{\partial^2\phi}{\partial y\partial z}-\frac{\partial^2\phi}{\partial z\partial y}\right)\underline{i}+\left(\frac{\partial^2\phi}{\partial z\partial x}-\frac{\partial^2\phi}{\partial x\partial z}\right)\underline{j}+\left(\frac{\partial^2\phi}{\partial x\partial y}-\frac{\partial^2\phi}{\partial y\partial x}\right)\underline{k}.$$
$$= \underline{0}$$

$\underline{\nabla}\times\underline{\nabla}\phi = 0$ $\quad\underline{\nabla}\cdot\underline{\nabla}\phi = \nabla^2\phi$ $\quad\underline{\nabla}\cdot(\underline{\nabla}\times\underline{f}) = ?$

Example: $\underline{F} = x^2 e^y \underline{i} + x\ln z \underline{j} + (x+z)\underline{k}$

$$\underline{G} = \nabla \times \underline{F} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2e^y & x\ln z & x+z \end{vmatrix} = -\frac{x}{z}\underline{i} - \underline{j} + (\ln z - x^2 e^y)\underline{k}$$

$$\underline{\nabla} \cdot \underline{G} = \frac{\partial G_1}{\partial x} + \frac{\partial G_2}{\partial y} + \frac{\partial G_3}{\partial z} = -\frac{1}{z} + 0 + \frac{1}{z} = 0$$

Theorem: $\underline{\nabla} \cdot (\underline{\nabla} \times \underline{F}) = 0$.