Properties of Grad

1) $\underline{\nabla}(\phi+\psi) = \underline{\nabla}\phi + \underline{\nabla}\psi$

Proof: Use Cartesian coordinates $\frac{\partial}{\partial x}(\phi+\psi) = \frac{\partial\phi}{\partial x} + \frac{\partial\psi}{\partial x}$ etc.

2) $\underline{\nabla}(\phi\psi) = \psi\underline{\nabla}\phi + \phi\underline{\nabla}\psi$ Leibniz Law

Proof: Use Cartesians $\frac{\partial}{\partial x}(\phi\psi) = \psi\frac{\partial\phi}{\partial x} + \phi\frac{\partial\psi}{\partial x}$. $((fg)' = fg' + f'g)$.

Example: Let $\underline{r} = x\underline{i} + y\underline{j} + z\underline{k}$. Consider the vector field $\phi\underline{r} = |\underline{r}| = r$

What is $\underline{\nabla}\phi$?

Cartesian Method: $\phi(x,y,z) = (x^2+y^2+z^2)^{1/2}$ $\quad \frac{\partial\phi}{\partial x} = 2x\cdot\frac{1}{2}(x^2+y^2+z^2)^{-1/2}$

$= \frac{x}{(x^2+y^2+z^2)^{1/2}} = \frac{x}{r}$

$\frac{\partial\phi}{\partial x} = \frac{x}{r} \quad \frac{\partial\phi}{\partial y} = \frac{y}{r} \quad \frac{\partial\phi}{\partial z} = \frac{z}{r}$

$\underline{\nabla}\phi = \frac{\partial\phi}{\partial x}\underline{i} + \frac{\partial\phi}{\partial y}\underline{j} + \frac{\partial\phi}{\partial z}\underline{k} = \frac{x}{r}\underline{i} + \frac{y}{r}\underline{j} + \frac{z}{r}\underline{k} = \frac{x\underline{i}+y\underline{j}+z\underline{k}}{r} = \frac{\underline{r}}{r} = \hat{\underline{r}}$

Geometric method: Level surfaces $r = $ constant

$\phi = $ constant

$\underline{\nabla}\phi = \frac{\partial\phi}{\partial n}\hat{\underline{n}}$

$= \frac{\partial\phi}{\partial r}\hat{\underline{r}}$

$= 1\cdot\hat{\underline{r}}$

Scalar field $\xrightarrow{\text{GRAD}}$ Vector field

Vector field $\xrightarrow{\text{DIV}}$ Scalar field.

Divergence of a Vector field.

Cartesian definition Let $\underline{F}(\underline{r})$ be a differentiable vector field

$\underline{F}(\underline{r}) = F_1(\underline{r})\underline{i} + F_2(\underline{r})\underline{j} + F_3(\underline{r})\underline{k}$

We define div $\underline{F}$ to be the scalar field given by $\text{div}\,\underline{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$

Example: Let $\underline{F} = xy\underline{i} + y^2\underline{j} + x^2yz\underline{k}$. Calculate div $\underline{F}$.

$(\text{div}\,\underline{F} = y + 2y + x^2y = 3y + x^2y)$

$\text{div}\,\underline{F} = \frac{\partial}{\partial x}(xy) + \frac{\partial}{\partial y}(y^2) + \frac{\partial}{\partial z}(x^2yz) = y + 2y + x^2y = 3y + x^2y$.

We can write this in a different way as

$\text{div}\,\underline{F} = (\underline{i}\frac{\partial}{\partial x} + \underline{j}\frac{\partial}{\partial y} + \underline{k}\frac{\partial}{\partial z})\cdot(F_1\underline{i} + F_2\underline{j} + F_3\underline{k})$.

$= \underline{\nabla}\cdot\underline{F}$ where $\underline{\nabla}$ is the differential operator given by

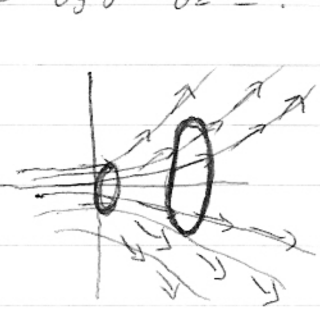
$\underline{\nabla} = \underline{i}\frac{\partial}{\partial x} + \underline{j}\frac{\partial}{\partial y} + \underline{k}\frac{\partial}{\partial z}$

Note that this is consistant with grad $\phi = \underline{\nabla}\phi = \frac{\partial\phi}{\partial x}\underline{i} + \frac{\partial\phi}{\partial y}\underline{j} + \frac{\partial\phi}{\partial z}\underline{k}$.

Look at the effect of a vector field on a small sphere

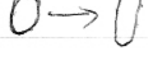
Flow lines are moving apart so a small sphere of fluid expands (or diverges).

This vector field has +ve divergence.



A vector field where the field lines move together has negative divergence.

Fields for which the volume of a small sphere remains constant are called divergence free and have div $\underline{F} = 0$.



Suppose we start with a scalar field ϕ. Then $\underline{\nabla}\phi$ is a vector field, so we can calculate its divergence: $\underline{\nabla}\cdot(\underline{\nabla}\phi)$. We write this as $\nabla^2\phi$ and call it the Laplacian of ϕ.

What is $\nabla^2\phi$ in Cartesian coordinates?

$$\underline{F} = \underline{\nabla}\phi = \frac{\partial\phi}{\partial x}\underline{i} + \frac{\partial\phi}{\partial y}\underline{j} + \frac{\partial\phi}{\partial z}\underline{k}$$

$$\text{div } \underline{F} = \underline{\nabla}\cdot\underline{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$$

$$\nabla^2\phi = \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2}$$