Gradients + Steepest Ascent.

$\underline{\nabla}\phi = \frac{\partial\phi}{\partial x}\underline{i} + \frac{\partial\phi}{\partial y}\underline{j} + \frac{\partial\phi}{\partial z}\underline{k}$

Given a scalar field $\phi(\underline{r})$ in what direction is it increasing most?
Let $\hat{\underline{v}}$ be a unit vector, then $\nabla_{\hat{v}}\phi = \hat{\underline{v}} \cdot \underline{\nabla}\phi = |\hat{\underline{v}}||\underline{\nabla}\phi|\cos\theta = \cos\theta|\underline{\nabla}\phi|$.
So this is a max when $\theta = 0$, min when $\theta = \pi$.

$$-|\underline{\nabla}\phi| \leq \nabla_{\hat{v}}\phi \leq |\underline{\nabla}\phi|$$

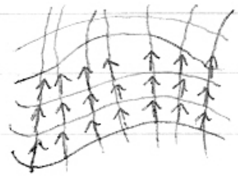
Theorem 1: The directional derivative $\nabla_{\hat{v}}\phi$ is maximised when $\hat{\underline{v}}$ points in the same direction of $\underline{\nabla}\phi$ and minimised when it points in the opposite direction. furthermore the max and min. values are $\pm|\underline{\nabla}\phi|$

Gradients and Level Surfaces.

We want to look at $\underline{\nabla}\phi$ at the point P with co-ordinates (x_0, y_0, z_0).
Let $\phi(x_0, y_0, z_0) = C_0$. Now consider the level surface through P.
This is $\phi(x, y, z) = C_0$.

Now look at the tangent plane to the surface at P.
Let $\underline{v}$ be a tangent vector at P. Then since $\underline{v}$ is tangent to the surface $\phi(x,y,z) = C_0 = $ Constant. The derivative of ϕ in $\underline{v}$ direction is zero.
Thus $\nabla_{\underline{v}}\phi = 0$ $\underline{v} \cdot \underline{\nabla}\phi = 0$. i.e. the tangent vector $\underline{v}$ is perpendicular to $\underline{\nabla}\phi$ $\underline{\nabla}\phi$ is perpendicular to the tangent plane so $\underline{\nabla}\phi$ is normal. to the surface $\phi = C_0$.

Theorem 2: $\underline{\nabla}\phi$ is normal to the surfaces $\phi = $ constant.



Since $\underline{\nabla}\phi$ is normal to the surface $\phi = $ constant
$\Rightarrow \underline{\nabla}\phi = k\hat{\underline{n}}$ $|\underline{\nabla}\phi| = k$ But we know from theorem 1 that $|\underline{\nabla}\phi| = \nabla_{\hat{n}}\phi$.
(where $\hat{\underline{n}}$ is taken in the direction of increasing ϕ).
Hence $\underline{\nabla}\phi = (\nabla_{\hat{\underline{n}}}\phi)\hat{\underline{n}}$. To simplify notation we write $\nabla_{\hat{\underline{n}}}\phi$ as $\frac{\partial\phi}{\partial n}$.

Hence $\underline{\nabla}\phi = \frac{\partial\phi}{\partial n}\hat{\underline{n}}$

Gradient (Geometric definition).

The gradient of the scalar field ϕ is $\underline{\nabla}\phi = \frac{\partial\phi}{\partial n}\hat{\underline{n}}$ where $\hat{\underline{n}}$ is the unit normal to the surfaces $\phi = $ constant and $\frac{\partial\phi}{\partial n}$ is the derivative of ϕ in the $\hat{\underline{n}}$ direction. Note: $\underline{\nabla}\phi$ is a vector field.

Gradient (Cartesian coordinate definition).

$$\underline{\nabla}\phi = \frac{\partial \phi}{\partial x}\underline{i} + \frac{\partial \phi}{\partial y}\underline{j} + \frac{\partial \phi}{\partial z}\underline{k}.$$

Direction derivative

Definition $\nabla_{\underline{v}}\phi(\underline{r}_0) = \lim_{t\to 0}\left\{\frac{\phi(\underline{r}_0 + t\underline{v}) - \phi(\underline{r}_0)}{t}\right\}$

$= \underline{v}\cdot\underline{\nabla}\phi$

$= v_1\frac{\partial \phi}{\partial x} + v_2\frac{\partial \phi}{\partial y} + v_3\frac{\partial \phi}{\partial z}$ (in Cartesians).

Example: Calculate the equation of the tangent plane to the surface S given by $x^3y - yz^2 + z^5 = 9$ at the point $(3, -1, 2) = \underline{r}_0$.

Let $\phi = x^3y - yz^2 + z^5$ $\quad \phi(\underline{r}_0) = 9$ ✓ so S is the level surface $\phi(\underline{r}) = 9$.

$\underline{\nabla}\phi = 3x^2y\,\underline{i} + (x^3 - z^2)\underline{j} + (5z^4 - 2yz)\underline{k}$

$\underline{\nabla}\phi(\underline{r}_0) = -27\underline{i} + 23\underline{j} + 84\underline{k} = \underline{n}$

Tangent plane: $\underline{n}\cdot(\underline{r} - \underline{r}_0) = 0.$

$(-27\underline{i} + 23\underline{j} + 84\underline{k})\cdot((x-3)\underline{i} + (y+1)\underline{j} + (z-2)\underline{k}) = 0.$

$-27x + 23y + 84z = 64.$