Friday 10TH October 2008 MATH2045 - Vector Calculus + Complex Variable Theory

Vector Fields

Example 1

$\underline{F}(x,y) = y\underline{i} - x\underline{j}$

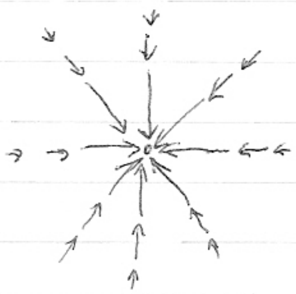
Example 2

Let $\underline{r} = x\underline{i} + y\underline{j}$

$\underline{F}(\underline{r}) = -k\dfrac{\underline{r}}{|\underline{r}|^3}$

$= -\dfrac{k}{|\underline{r}|^2}\hat{\underline{r}}$

inverse square law



$\underline{F}$ is a vector field on $\mathbb{R}^2 \setminus \{(0,0)\}$.

The flow lines (or integral curves) of a vector field are curves $\underline{x}(t)$ such that $\underbrace{\dot{\underline{x}}(t)}_{\text{tangent to the curve}} = \underbrace{\underline{F}(\underline{x}(t))}_{\text{vector field.}}$

In example 1 if $\underline{x}(t) = x(t)\underline{i} + y(t)\underline{j}$ $\quad \dfrac{d\underline{x}}{dt} = \dfrac{dx}{dt}\underline{i} + \dfrac{dy}{dt}\underline{j}$

$\dfrac{dx}{dt} = y \quad \dfrac{dy}{dt} = -x \quad \Big\} \dfrac{dy}{dx} = -\dfrac{x}{y} \quad \int y\,dy + \int x\,dx = \text{constant} \quad \frac{1}{2}(x^2+y^2) = C.$

Derivatives of scalar fields

Let $\phi(\underline{r})$ be a scalar field. What is the derivative? The change in ϕ depends on what direction you move in.

Directional derivative

Let $\underline{v}$ be a fixed vector. Then the directional derivative of ϕ in $\underline{v}$ direction at the point $\underline{r}$ is $\quad \nabla_{\underline{v}}\phi(\underline{r}) = \lim_{t\to 0}\left\{\dfrac{\phi(\underline{r}+t\underline{v}) - \phi(\underline{r})}{t}\right\}.$

Definition of a derivative: $\dfrac{df}{dx} = \lim_{t\to 0}\left\{\dfrac{f(x+t)-f(x)}{t}\right\}$

This measures the change of ϕ in the $\underline{v}$ direction.

Think of $\underline{r}$ and $\underline{v}$ as fixed and define $f(t) = \phi(\underline{r}+t\underline{v})$ — ⊛

Then $\nabla_{\underline{v}}\phi(\underline{r}) = \lim_{t\to 0}\left\{\dfrac{f(t)-f(0)}{t}\right\} = f'(0)$

We can write ⊛ as $f(t) = \phi(x,y,z)$ — (1)

where $x = r_1 + tv_1 \quad y = r_2 + tv_2 \quad z = r_3 + tv_3 \quad \underline{r} = r_1\underline{i} + r_2\underline{j} + r_3\underline{k} \quad \underline{v} = v_1\underline{i} + v_2\underline{j} + v_3\underline{k}$

Differentiate (1) wrt t using the chain rule:

$$\frac{df}{dt} = \frac{\partial\phi}{\partial x}\frac{dx}{dt} + \frac{\partial\phi}{\partial y}\frac{dy}{dt} + \frac{\partial\phi}{\partial z}\frac{dz}{dt} = v_1\frac{\partial\phi}{\partial x} + v_2\frac{\partial\phi}{\partial y} + v_3\frac{\partial\phi}{\partial z}$$

So introduce $\nabla\phi = \dfrac{\partial\phi}{\partial x}\underline{i} + \dfrac{\partial\phi}{\partial y}\underline{j} + \dfrac{\partial\phi}{\partial z}\underline{k}$

then $\dfrac{df}{dt}(0) = \underline{v}\cdot\nabla\phi(\underline{r})$

Hence $\nabla_{\underline{v}}\phi(\underline{r}) = \underline{v}\cdot\nabla\phi(\underline{r})$. $\quad$ -grad ϕ

Example $\phi(x,y,z) = xy + z^2$

What is the rate of change of ϕ in the direction $\hat{\underline{v}}$ at the point $x=1,\ y=2,\ z=1$ where $\underline{v} = 3\underline{i} + 4\underline{k}$

$$\nabla\phi = \frac{\partial\phi}{\partial x}\underline{i} + \frac{\partial\phi}{\partial y}\underline{j} + \frac{\partial\phi}{\partial z}\underline{k} = y\underline{i} + x\underline{j} + 2z\underline{k}.$$

$\hat{\underline{v}} = \frac{\underline{v}}{|\underline{v}|}$ $\quad |\underline{v}| = \sqrt{9+16} = 5.$ $\quad \hat{\underline{v}} = \frac{3}{5}\underline{i} + \frac{4}{5}\underline{k}$

$$\nabla_{\hat{\underline{v}}}\phi(1,2,1) = \hat{\underline{v}} \cdot \nabla\phi = \left(\tfrac{3}{5}\underline{i} + \tfrac{4}{5}\underline{k}\right)\cdot(2\underline{i} + \underline{j} + 2\underline{k}) = \frac{14}{5}.$$