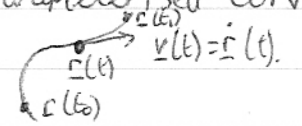
Curves.

Parameterised curve is a map $\underline{r}: \mathbb{R} \to \mathbb{R}^3 \quad t \mapsto \underline{r}(t)$



Speed: $v(t) = |\dot{\underline{r}}(t)|$

distance $= \int_{t=t_0}^{t_1} v(t)\,dt = \int_{t=t_0}^{t_1} \left|\frac{d\underline{r}}{dt}\right| dt.$

What happens if we use some other parameterisation of the curve

$t = t(\lambda), \ \lambda = \lambda(t)$ Then $\frac{d\underline{r}}{dt} = \frac{d\underline{r}}{d\lambda}\frac{d\lambda}{dt} \quad dt = \frac{dt}{d\lambda}d\lambda$

monotonic function - increasing.

distance can also be written $\int_{\lambda=\lambda_0}^{\lambda_1} \left|\frac{d\underline{r}}{d\lambda}\frac{d\lambda}{dt}\right| \frac{dt}{d\lambda}\,d\lambda$

with $\lambda_0 = \lambda(t_0) \quad \lambda_1 = \lambda(t_1)$

$= \int_{\lambda=\lambda_0}^{\lambda_1} \left|\frac{d\underline{r}}{d\lambda}\right| \frac{d\lambda}{dt}\frac{dt}{d\lambda}\,d\lambda$

$= \int_{\lambda=\lambda_0}^{\lambda_1} \left|\frac{d\underline{r}}{d\lambda}\right| d\lambda$

reparameterisation invariance.

Scalar fields

A scalar field is a scalar valued function that depends upon position e.g. the temperature in this room.

In 2 dimensions it is a map $\phi: \mathbb{R}^2 \to \mathbb{R} \quad (x,y) \mapsto \phi(x,y)$.

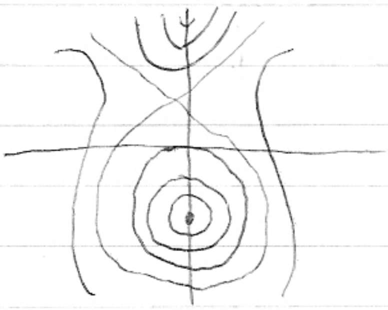
for example $\phi(x,y) = \frac{1}{12}y^3 - y - \frac{1}{4}x^2 + \frac{7}{2}$

To visualise this we can plot $z = \phi(x,y)$ using MAPLE for example.

An alternative is to plot the contour lines or level curves given by $z =$ constant

$\phi(x\,y) = c \qquad \frac{1}{12}y^3 - y - \frac{1}{4}x^2 + \frac{7}{2} = c$

rather than plot a single contour, we plot a number of different contours at various equally spaced heights.



Example: plot the level curves of

$f(x,y) = 4 - x^2 - y^2$

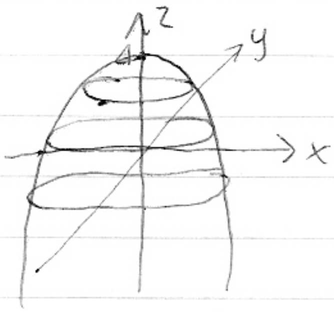
$4 - x^2 - y^2 = c \implies x^2 + y^2 = 4 - c$

circle, centre origin, radius $\sqrt{4-c}$ Note $c \le 4$.

$c=4 \quad x^2+y^2=0$ ·

$c=3 \quad x^2+y^2=1$ ○

$c=2 \quad x^2+y^2=2$ ○ →



Now consider scalar functions in $\mathbb{R}^3$

$\phi : \mathbb{R}^3 \to \mathbb{R}$

$(x,y,z) \mapsto \phi(x,y,z)$

Graph the function $w = \phi(x,y,z)$

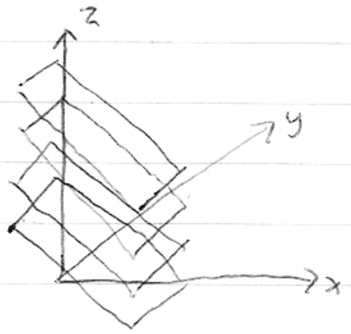
3-dimensional hypersurface in 4-dimensional space

rather hard to draw!

Instead we draw the level surfaces

$\phi(x,y,z) = c = \text{const}$

2-dim surface in $\mathbb{R}^3$.



Example: Let $\phi(x,y,z) = x+y+z$

Level surfaces are $x+y+z=c$

planes with normal $\underline{n} = \underline{i}+\underline{j}+\underline{k}$

$\underline{n} = \underline{i}+\underline{j}+\underline{k} \quad \underline{r} = x\underline{i}+y\underline{j}+z\underline{k}$

$\underline{n}\cdot(\underline{r}-\underline{r}_0)=0 \quad \underline{n}\cdot\underline{r} = \underline{n}\cdot\underline{r}_0 = c.$

Vector Fields

A vector field is a vector valued function of position

In 2 dimensions it is a map $\underline{F} : \mathbb{R}^2 \to \mathbb{R}^2 \quad (x,y) \mapsto \underline{f}(x,y) = f_1(x,y)\underline{i} + f_2(x,y)\underline{j}$

Example 1: $\underline{F}(x,y) = y\underline{i} - x\underline{j}$

How do we depict $\underline{F}$? At the point (x,y) draw the vector $\underline{f}(x,y)$