Bolzano – Weierstrass.

Definition: A set S is sequentially compact if for every sequence $x_n \in S$ there is a subsequence x_{n_k} s.t. $x_{n_k} \to L$ with $L \in S$.

Bolzano - Weierstrass Theorem: $[a,b]$ the closed bounded interval is sequentially compact.

Examples 1) $S = [a,b]$ is sequentially compact.

2) $S = \mathbb{R}$ is NOT sequentially compact.

Let $x_n = n \in S$ Every subsequence $\to \infty$ No subsequence converges to $L \in S$.

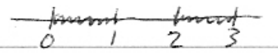
3) $S = [0,1) = \{x \in \mathbb{R} : 0 \leq x < 1\}$. This is <u>NOT</u> sequentially compact. Why?

for example: $x_n = 1 - \frac{1}{n}$ $n = 1, 2, 3 \ldots$ $x_n \in S$ $x_n \to 1$ as $n \to \infty$ $1 \notin S$.

As x_n converges, every subsequence x_{n_k} also converges to $1 \notin S$.

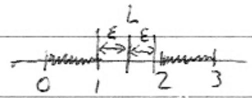
No subsequence converges to a limit in S.

4) $S = [0,1] \cup [2,3]$. This is sequentially compact



Prove this: Let x_n be a sequence in S. Then x_n is bounded

$x_n \in [0,3]$ for all n so by B-W theorem $\exists$ subsequence x_{n_k} of x_n s.t. x_{n_k} converges to $L \in [0,3]$

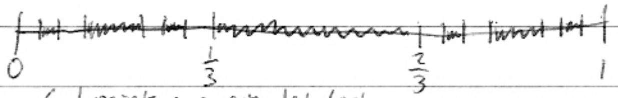


Suppose $L \notin S$ i.e. $L \in (1,2)$.

Let $\varepsilon = \min\{2-L, L-1\}$ then if x with $L - \varepsilon < x < L + \varepsilon$ then $1 < x < 2$ since $x_{n_k} \to L$ as $k \to \infty$ there exists k_0 s.t. $k > k_0$ then $|x_{n_k} - L| < \varepsilon$, so $L - \varepsilon < x_{n_k} < L + \varepsilon$

So $1 < x_{n_k} < 2$. This is a contradiction because $x_{n_k} \in S$.

Exercise: S = Cantor set



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Question: is S sequentially compact?

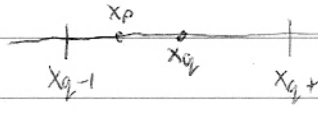
General principle of convergence (GP of C): Every Cauchy sequence in $\mathbb{R}$ converges.

Lemma: If $x_n \in \mathbb{R}$ is Cauchy then x_n is bounded.

Proof: We know $\forall \varepsilon > 0$ $\exists n_0$ s.t. $\forall p, q$ if $p, q > n_0 \Rightarrow |x_p - x_q| < \varepsilon$.

Pick $\varepsilon = 1$ Then $\exists n_0$ s.t. if $p, q > n_0$ then $|x_p - x_q| < 1$.

$\Rightarrow -1 < x_p - x_q < 1$ so $x_q - 1 < x_p < x_q + 1$. fix $q > n_0$ then for every $p > n_0$ we have x_p is bounded.



Let $a = \min\{x_1, x_2, \dots, x_{n_0}, x_q - 1\}$. Let $b = \max\{x_1, x_2, \dots, x_{n_0}, x_q + 1\}$.
For any p we have either $p \leq n_0$, so $a \leq x_p \leq b$ or $p > n_0$ so
$x_q - 1 < x_p < x_q + 1$ $a \leq x_q - 1$ so $a \leq x_p$ and $x_q + 1 \leq b$, so $x_p \leq b$.
Either way $x_p \in [a, b]$ so x_p is bounded. $\square$.

Theorem (GP of C): Suppose x_n is a Cauchy sequence in $\mathbb{R}$. Then x_n is convergent.
Proof: By the lemma x_n is bounded so $x_n \in [a, b]$ for some $a, b \in \mathbb{R}$.
Thus, by the B-W theorem, there is some subsequence which converges:
x_{n_k} a subsequence s.t. $x_{n_k} \to L \in [a, b]$
We want to prove that $x_n \to L$, so we need
$\forall \varepsilon > 0 \ \exists N_0$ s.t. $\forall n \ n > N_0 \Rightarrow |x_n - L| < \varepsilon$
We know x_n is Cauchy, thus, given any $\varepsilon > 0 \ \exists N_0$ s.t. if $p, q > N_0$
then $|x_p - x_q| < \varepsilon/2$. Also, we know that $x_{n_k} \to L$ so $\exists k_0$ s.t.
$k > k_0 \Rightarrow |x_{n_k} - L| < \varepsilon/2$
Pick $k > k_0$ s.t. $n_k > N_0$. Since $k > k_0$ we know $|x_{n_k} - L| < \frac{\varepsilon}{2}$
Since $n_k > N_0$ we know (setting $q = n_k > N_0$) that $\forall p > N_0 \ |x_p - x_{n_k}| < \frac{\varepsilon}{2}$.
Thus for any $p > N_0$ we have $|x_p - L| \leq |x_p - x_{n_k}| + |x_{n_k} - L| < \varepsilon$
(by triangle inequality) $\square$.