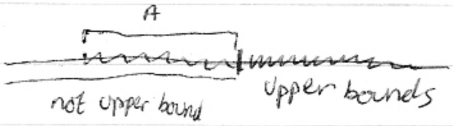
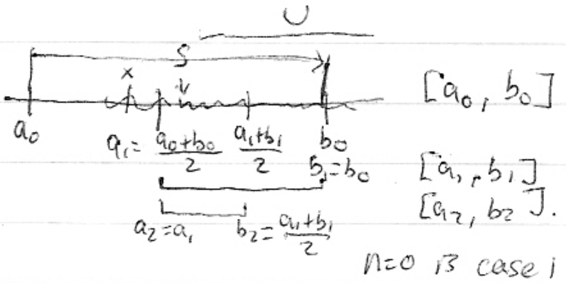
Tuesday 28th October 2008 MATH2039 - Analysis

$A \subseteq \mathbb{R}$ non-empty, bounded above



Theorem $A \subseteq \mathbb{R}$ non-empty bounded above the A has a supremum
(similar for infimum)

Let U = upper bounds of A
N = non upper-bounds = $\mathbb{R} \setminus U$
pick $a_0 \in N$ $b_0 \in U$



proceed by induction.
Given a_n, b_n consider $\frac{a_n+b_n}{2}$.

Case 1: $\frac{a_n+b_n}{2} \in N$ Then let $a_{n+1} = \frac{a_n+b_n}{2}$, let $b_{n+1} = b_n$

Case 2: $\frac{a_n+b_n}{2} \in U$ Then let $a_{n+1} = a_n$, let $b_{n+1} = \frac{a_n+b_n}{2}$.

We produce a sequence a_n which is non-decreasing ($a_n \le a_{n+1}$)
and a sequence b_n which is non-increasing ($b_n \geq b_{n+1}$)

All the terms a_n lie between a_0 and b_0 so a_n is bounded. It is also monotonic, so it converges. All the b_n lie between a_0 and b_0 so b_n is bounded and monotonic so it converges.

The difference $b_n - a_n$ is halved at each stage so $b_n - a_n = \frac{b_0 - a_0}{2^n} \to 0$.

Let $S = \lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n$

Check S is the supremum: 1) $\forall x \in A$ $x \le S$
2) If $u \in \mathbb{R}$ $\forall x \in A$ $u \geq x$ then $u \geq S$ (If $u \in U$ then $u \geq S$).

1) Take any $x \in A$ then $x \le b_n$ for all n because $b_n \in U$. Thus $x \le \lim_{n\to\infty} b_n = S$

2). Let u be any upper bound $u \in U$. Then, by construction, each $a_n \in N$, so $a_n \le u$ so $\lim_{n\to\infty} a_n \le u$ Thus $S \le u$ □.

Bolzano-Weierstrass Theorem.

Example: Which of these sequences have convergent subsequences.

1) $a_n = (-1)^n \left(\frac{n+1}{n}\right)$ $-\frac{2}{1}, +\frac{3}{2}, -\frac{4}{3}, +\frac{5}{4}, \ldots$ ← bounded.

2) $b_n = n$ $1, 2, 3, \ldots$ ← unbounded.

1) a_n has (at least) 2 convergent subsequences.

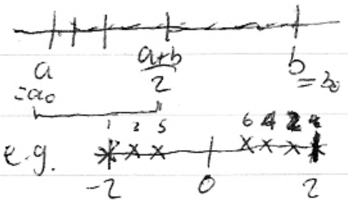
$a_1, a_3, a_5, \ldots$ $a_{n_k} = a_{2k+1}$ odd terms $a_{2k+1} = (-1)^{2k+1}\left(\frac{2k+1+1}{2k+1}\right)$

$= -\frac{2k+2}{2k+1} \to -1$ as $k \to \infty$

$a_2, a_4, a_6, \ldots \quad a_{n_k} = a_{2k} \quad a_{2k} = (-1)^{2k} \frac{2k+1}{2k} = \frac{2k+1}{2k} \to 1$

2) Every subsequence diverges.

Theorem (Bolzano-Weierstrass): Let $x_n \in \mathbb{R}$ and suppose $x_n \in [a,b]$ for all n. (x_n is bounded). Then x_n has (at least) one convergent subsequence x_{n_k} s.t. $x_{n_k} \to L \in [a,b]$



Proof: Let $a_0 = a$ and $b_0 = b$. Consider the intervals $[a_0, \frac{a_0+b_0}{2}]$ and $[\frac{a_0+b_0}{2}, b_0]$.

we ask how many of the terms x_n lies in each of these?

If $x_n \in [a_0, \frac{a_0+b_0}{2}]$ for infinitely many n then choose this half.

Let $a_1 = a_0$ and $b_1 = \frac{a_0+b_0}{2}$.

otherwise $x_n \in [\frac{a_0+b_0}{2}, b_0]$ for infinitely many n. So let $a_1 = \frac{a_0+b_0}{2}$, $b_1 = b_0$.

Repeat this to produce a_k, b_k for $k = 1, 2, 3, \ldots$ such that for each k we have infinitely many n s.t. $x_n \in [a_k, b_k]$.

As in the previous theorem a_k and b_k are bounded and monotonic.

$b_k - a_k = \frac{b-a}{2^k} \to 0$ as $k \to 0$ let $L = \lim_{k\to\infty} a_k = \lim_{k\to\infty} b_k$.

for each k pick $x_{n_k} \in [a_k, b_k]$ with $n_k > n_{k-1}$ (there are infinitely many of these).

for each k $a_k \leq x_{n_k} \leq b_k$ $a_n \to L$ and $b_k \to L$

So $x_{n_k} \to L$ by squeeze rule $\square$