Monday 27TH October 2008 MATH2039 - Analysis

Infimum and Supremum

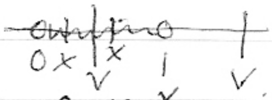
Example $A = (0,1) = \{x \in \mathbb{R} : 0 < x < 1\}$.

Show that $\inf A = 0$. Check: 1) $\forall x \in A \quad 0 \leq x$

2) If $V \in \mathbb{R}$ s.t. $\forall x \in A \quad V \leq x$ then $V \leq 0$.

1) By definition if $x \in A$ then $0 < x$, so $0 \leq x$.

2) Suppose $V \in \mathbb{R}$ and $\forall x \in A \quad V \leq x$. Also suppose $V > 0$



we can find x with $0 < x < V$ moreover we can find x with $x < 1$

eg. $x = \frac{V}{2}$

eg. $x = \frac{(\min(V,1))}{2}$

$0 < x < 1 \Rightarrow x \in A$ and $x < V$ but we supposed $\forall x \in A \quad x \geq V$ which is a contradiction so $V \leq 0$.

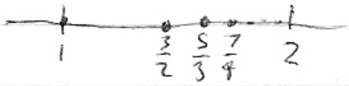
Example $A = \{\frac{2n-1}{n} ; n = 1, 2, 3, \ldots\} = \{\frac{1}{1}, \frac{3}{2}, \frac{5}{3}, \frac{7}{4}, \frac{9}{5}, \ldots\}$.

What is the inf and sup? $\frac{2n-1}{n}$ is increasing

$\frac{2n-1}{n} = 2 - \frac{1}{n}$ and $\frac{1}{n}$ is decreasing thus $\frac{2n-1}{n} \geq \frac{2 \cdot 1 - 1}{1} = 1$ for all n.

This is 1) $\forall x \in A \quad 1 \leq x$ we also need 2) If $V \in \mathbb{R}$ s.t. $\forall x \in A \quad V \leq x$ then $V \leq 1$. By assumption $V \leq x$ for every $x \in A$ but $1 \in A$ so $V \leq 1$.

What is the supremum? we'll show $\sup A = 2$



We need 1) $\forall x \in A \quad x \leq 2$

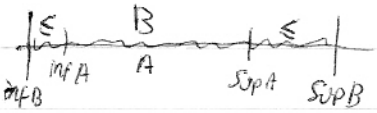
2) If $U \in \mathbb{R} \quad \forall x \in A \quad x \leq U$ then $2 \leq U$

for 1) If $x \in A$ then $x = \frac{2n-1}{n} = 2 - \frac{1}{n} < 2$ so $x \leq 2$

2) Suppose $U \in \mathbb{R}$ s.t. $\forall x \in A \quad x \leq U$. In other words $\forall n \in \mathbb{N}$ $\frac{2n-1}{n} \leq U$. Thus by inequalities of limits $\lim_{n \to \infty} \frac{2n-1}{n} \leq U$.

$\lim_{n \to \infty} \frac{2n-1}{n} = 2$, so $2 \leq U$.

Proposition. If $A \subseteq B \subseteq \mathbb{R}$ with A non-empty and B bounded then $\sup A \leq \sup B$ and $\inf A \geq \inf B$.



Proof. Let $S_A = \sup A$ and $S_B = \sup B$

We know **1A)** $\forall x \in A \quad x \leq S_A$ **2A)** If $U_A \in \mathbb{R}$ s.t. $\forall x \in A \quad x \leq U_A$ then $S_A \leq U_A$ **1B)** $\forall x \in B \quad x \leq S_B$. **2B)** If $U_B \in \mathbb{R}$ s.t. $\forall x \in B$ $x \leq U_B$ then $S_B \leq U_B$.

If $x \in A$ then $x \in B$ because $A \subseteq B$ so $\forall x \in A \quad x \leq S_B$ (by 1B)

Now let $U_A = S_B$ $U_A \in \mathbb{R}$ and $\forall x \in A \quad x \leq U_A$ so (by 2A) $S_A \leq U_A$ thus $S_A \leq S_B$ $\square$. (infimum follows with opposite inequalities).

Proposition If A is non-empty, bounded and S_1 and S_2 are both Supremums of A then $S_1 = S_2$. Likewise for $\inf A$.

Proof: Since S_1 is a supremum for A then $\forall x \in A \ x \leq S_1$ (S_1 is an upper bound for A). Let $U = S_1$. Since S_2 is a supremum and $\forall x \in A \ x \leq U = S_1$ we have $S_2 \leq U = S_1$.

We have shown $S_2 \leq S_1$, reversing the roles of S_1 and S_2 $\forall x \in A \ x \leq S_2$ (S_2 is an upper bound). Letting $U = S_2$ we get: $\forall x \in A \ x \leq U = S_2$ so $S_1 \leq U = S_2$.

$S_2 \leq S_1$ and $S_1 \leq S_2 \Rightarrow S_1 = S_2$.

Theorem: If $A \subseteq \mathbb{R}$ non-empty and bounded above then A has a (unique) supremum.

If $A \subseteq \mathbb{R}$ non-empty and bounded below then A has a (unique) infimum.

Let A be non-empty and bounded above. Then for every $x \in \mathbb{R}$ either x is an upper bound or x is not an upper bound. Suppose x is an upper bound and y is not an upper bound.

Since x is an upper bound we know $\forall z \in A \ z \leq x$.

Since y is not an upper bound we know $\exists z \in A$ s.t. $y < z$.

There is some $z \in A$ s.t. $y < z$, and for every z (including this one) we have $z \leq x$. Thus $y < x$.

Every upper bound is $>$ every non-upper bound.