Thursday 23rd October 2008 MATH 2039 - Analysis 3.

Sequences

Example a_n = formula $a_0 = 1$ $a_1 = \sqrt{2a_0} = \sqrt{2}$ $a_2 = \sqrt{2a_1} = \sqrt{2\sqrt{2}}$

$a_{n+1} = \sqrt{2a_n}$ $a_n = \sqrt{2\cdots\sqrt{2\sqrt{2}}}$ · n 2's No general formula for a_n

Show a_n converges and find the limit

~~using~~ Use the axiom: If we show a_n is bounded and monotonic, then a_n converges.

$1 < \sqrt{2} \Rightarrow 2 < 2\sqrt{2} \Rightarrow \sqrt{2} < \sqrt{2\sqrt{2}} \Rightarrow 2\sqrt{2} < 2\sqrt{2\sqrt{2}} \Rightarrow \sqrt{2\sqrt{2}} < \sqrt{2\sqrt{2\sqrt{2}}}$

$a_0 < a_1 \Rightarrow a_1 < a_2 \Rightarrow a_2 < a_3$.

If $a_{n-1} < a_n$ then we show that $a_n < a_{n+1}$ $a_{n-1} < a_n$

$\Rightarrow 2a_{n-1} < 2a_n \Rightarrow \sqrt{2a_{n-1}} < \sqrt{2a_n} \Rightarrow a_n < a_{n+1}$

So by induction we get $a_n < a_{n+1}$ $\forall n$. Thus a_n is increasing.

$\forall n$ $1 \le a_n$ Guess a_n is always < 100

· Check this $1 < 100 \Rightarrow 2 < 200 \Rightarrow \sqrt{2} < \sqrt{200} \approx 14.1\ldots < 100$

$\sqrt{2} < 100 \Rightarrow 2\sqrt{2} < 200 \Rightarrow \sqrt{2\sqrt{2}} < \sqrt{200} < 100$.

If $a_n < 100$ then $2a_n < 200$ so $a_{n+1} = \sqrt{2a_n} < \sqrt{200} < 100$

So by induction $a_n < 100$ for all n. We have shown a_n is increasing (monotonic) and $1 \le a_n < 100$ for all n, so bounded. Thus a_n converges.

Example: Consider the sequence a_{n+1} $a_n \to L$ some limit $a_{n+1} \to L$ because it is a subsequence $a_{n+1} = \sqrt{2a_n}$ $a_n \to L \Rightarrow 2a_n \to 2L \Rightarrow \sqrt{2a_n} \to \sqrt{2L}$

$a_{n+1} \to L$ and $a_{n+1} \to \sqrt{2L}$ So $L = \sqrt{2L}$ $\Rightarrow L^2 = 2L = L(L-2) = 0$

$\Rightarrow L = 0$ or $L = 2$. $a_n \ge 1$ for all n so $L = \lim_{n\to\infty} a_n \ge 1$ Thus $a_n \to 2$.

Infimum and Supremum

Definition: Let $A \subseteq \mathbb{R}$ An upper bound of A is $U \in \mathbb{R}$ such that $\forall a \in A$ $a \le U$

A lower bound of A is $V \in \mathbb{R}$ s.t. $\forall a \in A$ $V \le a$.

Example: $A = \{1, 2, 3, 4, 5\}$. Then 5 is an upper bound of A

1000000 is also an upper bound.

1 is a lower bound, so is 0, so is -1000000.

Example: $A = [0, \infty) = \{x \in \mathbb{R} : 0 \le x < \infty\}$ 0, -1, etc. are lower bounds

There is no upper bound.

Definition. $A \subseteq \mathbb{R}$ is bounded above if there are any upper bounds, bounded below if there are lower bounds, and bounded if there are upper and lower bounds.

e.g. $[0, \infty)$ is bounded below but not above

$\{1, 2, 3, 4, 5\}$ is bounded.

Definition: The infimum of A is $t \in \mathbb{R}$ if: 1) $\forall a \in A$ $t \leq a$

2) Suppose $v \in \mathbb{R}$ s.t. $\forall a \in A$ $v \leq a$ then $v \leq t$

The supremum of A is $s \in \mathbb{R}$ if: 1) $\forall a \in A$ $s \geq a$

2) Suppose $u \in \mathbb{R}$ s.t. $\forall a \in A$ $u \geq a$ then $u \geq s$.

Example: $A = \{1, 2, 3, 4, 5\}$ Show 1 is the infimum of A and 5 is the supremum of A

1 is infimum: check 1) $\forall a \in A$ $a \geq 1$ If $a \in A$ then $a = 1, 2, 3, 4,$ or 5.

so $a \geq 1$ $1 \geq 1$ $2 \geq 1$ $3 \geq 1$ $4 \geq 1$ $5 \geq 1$

2) let $v \in \mathbb{R}$ such that $\forall a \in A$ $v \leq a$ we need to check $v \leq 1$

As $v \leq a$ for every $a \in A$, in particular $v \leq 1$ because $1 \in A$.

So 1 is the infimum.

5 is supremum: Check 1) $\forall a \in A$ $5 \geq a$ This is obvious $5 \geq 5, 4, 3, 2, 1$

so $5 \geq a$ $\forall a \in A$.

2) Let $u \in \mathbb{R}$ s.t. $\forall a \in A$ $u \geq a$. Then $u \geq 5$ because $5 \in A$.

from now on write $\inf A$ = infimum of A $\sup A$ = supremum of A.

Reminder of interval notation

$[a, b] = \{x \in \mathbb{R} : a \leq x \leq b\}$ $(a, b) = \{x \in \mathbb{R} : a < x < b\}$.

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Example $A = (0, 1) = \{x \in \mathbb{R} : 0 < x < 1\}$

This set has no minimum or maximum

Show $\sup A = 1$ Check 1) $\forall x \in A$ $1 \geq x$ since $x \in A$ we have $1 > x$, so $1 \geq x$.

2) Suppose $u \in \mathbb{R}$ s.t. $\forall x \in A$ $u \geq x$ then $u \geq 1$.

Suppose $u \geq x$ $\forall x \in A$ but $u < 1$ then we can find a point x with $u < x < 1$ with $x > 0$. This is a contradiction, because $0 < x < 1$

So $x \in A$, but $x > u$ Thus we deduce $u \geq 1$.