Tuesday 21st October 2008 MATH2039 - Analysis

Example $a_n = \frac{1}{10n^2}$ $b_n = \frac{1}{n^3}$ $c_n = a_n + b_n = \frac{1}{10n^2} + \frac{1}{n^3}$

How big does n need to be, so that $|c_n - 0| < \varepsilon = \frac{1}{100} = 0.01$

i.e. $c_n < \frac{1}{100} = 0.01$?

	n	$\frac{1}{10n^2}$ a_n	$\frac{1}{n^3}$ b_n	$\frac{1}{10n^2} + \frac{1}{n^3}$ c_n
	$\vdots$			
	3	$\frac{1}{90}$	$\frac{1}{27}$	
$a_n < \varepsilon$ $b_n, c_n > \varepsilon$	4	$\frac{1}{160}$	$\frac{1}{64}$	
$a_n < \frac{\varepsilon}{2} = \frac{1}{200}$ $b_n < \varepsilon$ $c_n > \varepsilon$	5	$\frac{1}{250}$	$\frac{1}{125}$	0.012
$a_n < \frac{1}{200}$ $b_n < \frac{1}{200}$ $c_n < \frac{1}{200} + \frac{1}{200} = \frac{1}{100}$ so $c_n < \varepsilon$	6	$\frac{1}{360}$	$\frac{1}{216}$	0.0074

for $n > 4$ we have $\frac{1}{10n^2} < \frac{1}{200}$

for $n > 5$ we have $\frac{1}{n^3} < \frac{1}{200}$

So for $n > 5$ $\frac{1}{10n^2} + \frac{1}{n^3} < \frac{1}{200} + \frac{1}{200} = \frac{1}{100}$

for $\varepsilon = 0.01$, let $n_0 = 5$

Now prove that $c_n \to 0$ as $n \to \infty$

To prove this, we must give a general recipe for how big n should be to make $c_n < \varepsilon$. We want $\frac{1}{10n^2} < \frac{\varepsilon}{2}$ $\sqrt{1/5\varepsilon} < n \Rightarrow \frac{2}{10\varepsilon} < n^2 \Rightarrow \frac{1}{10n^2} < \frac{\varepsilon}{2}$

If $n > \sqrt{1/5\varepsilon}$ then $\frac{1}{10n^2} < \frac{\varepsilon}{2}$

We also want $\frac{1}{n^3} < \frac{\varepsilon}{2}$ $\sqrt[3]{2/\varepsilon} < n \Rightarrow \frac{2}{\varepsilon} < n^3 \Rightarrow \frac{1}{n^3} < \frac{\varepsilon}{2}$

If $n > \sqrt[3]{2/\varepsilon}$ then $\frac{1}{n^3} < \frac{\varepsilon}{2}$ for $\varepsilon = \frac{1}{100}$ we need

$n > \sqrt{1/5\varepsilon} = \sqrt{1/\frac{1}{20}} = \sqrt{20} = 4.47$ and we need $n > \sqrt[3]{2/\varepsilon} = \sqrt[3]{200} = 5.8\ldots$

If $n >$ the larger of ($>$ both of) $\sqrt{1/5\varepsilon}$ and $\sqrt[3]{2/\varepsilon}$ then $c_n < \varepsilon$

Example let $a_n = \frac{3n^3 - 1}{4n^3 + 3n}$. Find the limit.

$a_n = \frac{3 - 1/n^3}{4 + 3/n^2}$ we know $\frac{1}{n} \to 0$ as $n \to \infty$

so $\frac{1}{n} \cdot \frac{1}{n} \to 0 \cdot 0 = 0$ $\frac{1}{n} \cdot \frac{1}{n} \cdot \frac{1}{n} \to 0$.

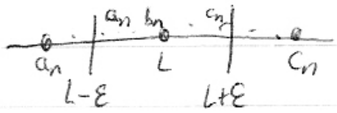
$\therefore a_n \to \frac{3 - 0}{4 + 3 \cdot 0} = \frac{3}{4}$.

Example $a_n = \frac{(-1)^n}{n}$ note that $-\frac{1}{n} \le \frac{(-1)^n}{n} \le \frac{1}{n}$ need theorem:

Theorem (squeeze rule / sandwich rule): If $a_n \le b_n \le c_n$ and $a_n \to L$ and $c_n \to L$ as $n \to \infty$ then $b_n \to L$ as $n \to \infty$.

Since $\frac{1}{n} \to 0$ and $-\frac{1}{n} \to -0 = 0$ we deduce that $\frac{(-1)^n}{n} \to 0$ as $n \to \infty$

Proof of Squeeze rule



We must show $\forall \varepsilon > 0$ $\exists n_0$ s.t.

$n > n_0 \Rightarrow |b_n - L| < \varepsilon$ $L - \varepsilon < b_n < L + \varepsilon$

We know that $a_n \to L$ as $n \to \infty$ so $\forall \varepsilon > 0$ $\exists n_1$ s.t. $n > n_1 \Rightarrow |a_n - L| < \varepsilon$

so that $L - \varepsilon < a_n < L + \varepsilon$

We also know $c_n \to L$ as $n \to \infty$ so $\forall \varepsilon > 0 \; \exists n_2$ s.t. $n > n_2 \Rightarrow |c_n - L| < \varepsilon$

so that $L - \varepsilon < c_n < L + \varepsilon$

Thus if $n >$ both n_1 and n_2 then we have $L - \varepsilon \leq a_n$ and $c_n < L + \varepsilon$

So $b_n \geq a_n > L - \varepsilon$ and $b_n \leq c_n < L + \varepsilon$

Thus $|b_n - L| < \varepsilon$

So we take n_0 to be whichever is greater of n_1, n_2 □

Example Show that $\frac{n + (-1)^n}{n} \to 1$ as $n \to \infty$

$-1 \leq (-1)^n \leq 1$ so $\frac{n-1}{n} \leq \frac{n + (-1)^n}{n} \leq \frac{n+1}{n}$ $\qquad \frac{n-1}{n} = 1 - \frac{1}{n} \to 1$

$\frac{n+1}{n} = 1 + \frac{1}{n} \to 1$ (arithmetic of sequences) so $\frac{n + (-1)^n}{n} \to 1$.

Example Find limit of $\frac{n^2}{2n^2 + \sin(n)}$

for all n we have $-1 \leq \sin(n) \leq 1$

so $2n^2 - 1 \leq 2n^2 + \sin(n) \leq 2n^2 + 1$

So $\frac{n^2}{2n^2 + \sin(n)} \leq \frac{n^2}{2n^2 - 1}$ and $\frac{n^2}{2n^2 + 1} \leq \frac{n^2}{2n^2 + \sin(n)}$

$\frac{n^2}{2n^2 - 1} = \frac{1}{2 - 1/n^2} \to \frac{1}{2 - 0} = \frac{1}{2}$ $\qquad \frac{n^2}{2n^2 + 1} = \frac{1}{2 + 1/n^2} \to \frac{1}{2 + 0} = \frac{1}{2}$.

Thus $\frac{n^2}{2n^2 + \sin(n)} \to \frac{1}{2}$ by the squeeze rule.