Monday 20th October 2008 MATH 2039 - Analysis

Sequences

Theorem (Arithmetic of sequences): Let $a_n : n \in I$, $b_n : n \in I$ be sequences. If $a_n \to a$ and $b_n \to b$ as $n \to \infty$ then

1) $a_n \pm b_n \to a \pm b$ as $n \to \infty$ 2) $a_n \cdot b_n \to a \cdot b$

3) If $b_n, b \neq 0$ then $\frac{1}{b_n} \to \frac{1}{b}$ 4) If $b_n, b \neq 0$ then $\frac{a_n}{b_n} \to \frac{a}{b}$

proof: 1) + case: Given $\varepsilon > 0$ we must find $n_0 \in I$ s.t. $n > n_0$ $\Rightarrow |(a_n + b_n) - (a+b)| < \varepsilon$

Rearrange: $|(a_n + b_n) - (a+b)| = |(a_n - a) + (b_n - b)| \leq |a_n - a| + |b_n - b|$ (triangle inequality) Idea: Make these $< \varepsilon/2$

We know $a_n \to a$ as $n \to \infty$, so given $\varepsilon > 0$ $\exists n_1 \in I$ s.t. $n > n_1 \Rightarrow |a_n - a| < \frac{\varepsilon}{2}$. We know $b_n \to b$ so $\exists n_2 \in I$ s.t. $n > n_2 \Rightarrow |b_n - b| < \frac{\varepsilon}{2}$ Choose $n_0 = \max\{n_1, n_2\}$.

If $n > n_0$ then $n > n_1 \Rightarrow |a_n - a| < \frac{\varepsilon}{2}$ and $n > n_2 \Rightarrow |b_n - b| < \frac{\varepsilon}{2}$

so $|(a_n + b_n) - (a+b)| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$

− case: Given $\varepsilon > 0$ we need $n_0 \in I$ s.t. $n > n_0 \Rightarrow$ $|(a_n - b_n) - (a-b)| < \varepsilon$. Rearrange: $|(a_n - b_n) - (a-b)| = |(a_n - a) - (b_n - b)|$ $\leq |a_n - a| + |-(b_n - b)| = |a_n - a| + |b_n - b|$

Take same n_0 as before, so that $|a_n - a| + |b_n - b| < \varepsilon$.

2) Given $\varepsilon > 0$ we want to find $n_0 \in I$ s.t. $n > n_0 \Rightarrow |a_n b_n - ab| < \varepsilon$.

Idea a_n close to a, so $a_n b_n$ close to ab_n and b_n close to b so ab_n close to ab. $|a_n b_n - ab| = |(a_n b_n - ab_n) + (ab_n - ab)|$

$\leq |a_n b_n - ab_n| + |ab_n - ab|$ make each term $< \frac{\varepsilon}{2}$

$|ab_n - ab| = |a(b_n - b)| = |a||b_n - b|$ as $b_n \to b$ we know $\exists n_1 \in I$ s.t. $n > n_1 \Rightarrow |b_n - b| < \frac{\varepsilon}{2|a|}$ if $a \neq 0$ and take any n_1 if $a = 0$

so if $n > n_1$ then $|a||b_n - b| < \frac{\varepsilon}{2}$

$|a_n b_n - ab_n| = |a_n - a||b_n|$ since b_n converges we know $|b_n|$ is bounded $\exists R > 0$ s.t. $|b_n| \leq R$ for all n.

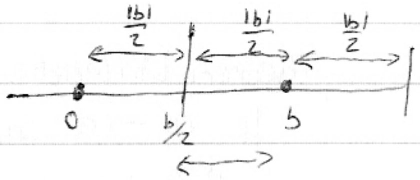
Thus $|a_n b_n - ab_n| \leq |a_n - a| R$. as $a_n \to a$ $\exists n_2 \in I$ s.t. $n > n_2 \Rightarrow |a_n - a| < \frac{\varepsilon}{2R} \Rightarrow |a_n - a| R < \frac{\varepsilon}{2} \Rightarrow |a_n b_n - ab_n| < \frac{\varepsilon}{2}$

Let $n_0 = \max\{n_1, n_2\}$ Then $n > n_0 \Rightarrow |a_n b_n - ab_n| < \frac{\varepsilon}{2}$ and $|ab_n - ab| < \frac{\varepsilon}{2}$. Putting this together $|a_n b_n - ab| < \varepsilon$.

3) Given $\varepsilon > 0$ we must find n_0 s.t. $n > n_0 \Rightarrow \left|\frac{1}{b_n} - \frac{1}{b}\right| < \varepsilon$

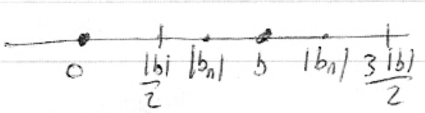
Rearranging: $\left|\frac{1}{b_n} - \frac{1}{b}\right| = \left|\frac{b - b_n}{b b_n}\right| = \frac{|b - b_n|}{|b||b_n|} = \frac{|b_n - b|}{|b|} \cdot \frac{1}{|b_n|}$

We'll prove $\frac{1}{|b_n|}$ is bounded



As $b_n \to b$ we know $\exists n_1 \in I$ s.t.
$n > n_1$ then $|b_n - b| < \frac{|b|}{2}$. This implies
$||b_n| - |b|| \le |b_n - b| < \frac{|b|}{2}$ (reverse triangle inequality)

for $n > n_1$ we have $|b| - \frac{|b|}{2} < |b_n| < |b| + \frac{|b|}{2}$



so $n > n_1 \Rightarrow |b_n| > \frac{|b|}{2} \Rightarrow \frac{1}{|b_n|} < \frac{2}{|b|}$
let $R = \max\left\{\frac{1}{|b_1|}, \frac{1}{|b_2|}, \dots, \frac{1}{|b_{n_1}|}, \frac{2}{|b|}\right\}$
then $\frac{1}{|b_n|} \le R \ \forall n$.
Thus for all n we have $\left|\frac{1}{b_n} - \frac{1}{b}\right| = \frac{|b_n - b|}{|b|} \cdot \frac{1}{|b_n|} \le \frac{|b_n - b|}{|b|} \cdot R$
Given $\varepsilon > 0$ $\exists n_0$ s.t. $n > n_0 \Rightarrow |b_n - b| < \frac{\varepsilon |b|}{R}$

$\Rightarrow \frac{|b_n - b| \cdot R}{|b|} < \varepsilon$ so $\left|\frac{1}{b_n} - \frac{1}{b}\right| < \varepsilon$

4). $\frac{a_n}{b_n} = a_n \cdot \frac{1}{b_n}$ by 3) $\frac{1}{b_n} \to \frac{1}{b}$ so by 2) $a_n \cdot \frac{1}{b_n} \to a \cdot \frac{1}{b} = \frac{a}{b}$ □