Thursday 16TH October 2008 MATH 2039 - Analysis 3

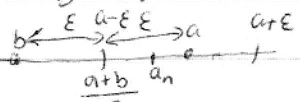
Sequences.

Definition: The integer part of x or floor of x is $\lfloor x \rfloor$ = greatest integer n s.t. $n \le x$. Note: this is always rounding down $\lfloor 1.5 \rfloor = 1$ $\lfloor -1.5 \rfloor = -2$.

Proposition: This is well-defined. That is $\forall x \in \mathbb{R}$ there is a greatest integer $m \in \mathbb{Z}$ s.t. $m \le x$

Proof: Let $S = \{n \in \mathbb{Z} : n \le x\}$ we must show this has a maximum / greatest element.
for any real number x $\exists$ integer $n_1 \in \mathbb{Z}$ s.t. $x < n_1$.
If $x \le 0$ then $n_1 = 1$ has this property. If $x > 0$ then $\exists n_1 \in \mathbb{Z}$ s.t. $n_1 > x$ by the Archimedean Property (for $1, x$)
Likewise $\exists n_0 \in \mathbb{Z}$ s.t. $n_0 > -x$. This implies $-n_0 < x$
Let $T = \{n \in S : n \ge -n_0\}$ If $n \in T$ then $n \in S$, so $n \le x < n_1$
Every $n \in T$ satisfies $-n_0 \le n < n_1$ so T is finite (has $\le n_0 + n_1$ elements)
Thus T has a maximum m. for any $n \in S$, either $n \in T$, so $n \le m$ or $n \notin T$ so $n < -n_0$, so $n \le m$ (because $-n_0 \le m$)
Thus m is a maximum of S. $\square$

Theorem (Inequalities of sequences): If $a_n \to a$ and $b_n \to b$ as $n \to \infty$ and $a_n \le b_n$ then $a \le b$.



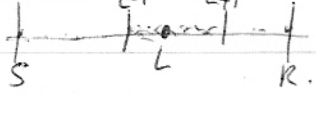
Proof: Suppose not, i.e. $a > b$. Choose $\varepsilon = \frac{a-b}{2} > 0$
Since $a_n \to a$ we know $\exists n_0 \in I$ s.t. $n > n_0 \Rightarrow |a_n - a| < \varepsilon$
$\Rightarrow a - \varepsilon < a_n < a + \varepsilon$

Note $a - \varepsilon = a - \frac{a-b}{2} = \frac{2a}{2} - \frac{a-b}{2} = \frac{a+b}{2}$
Since $b_n \to b$ $\exists n_1$ s.t. $n > n_1 \Rightarrow |b_n - b| < \varepsilon \Rightarrow b - \varepsilon < b_n < b + \varepsilon$.
Note $b + \varepsilon = b + \frac{a-b}{2} = \frac{2b + a - b}{2} = \frac{a+b}{2}$
Pick any n s.t. $n > n_0$ and $n > n_1$ then $a_n > a - \varepsilon = \frac{a+b}{2} = b + \varepsilon > b_n$
but we know $a_n \le b_n$, so this is a contradiction. Hence $a \le b$ $\square$.

Corollary: Limits are unique: If $a_n \to a$ and $a_n \to b$ as $n \to \infty$ then $a = b$.
Proof: Let $b_n = a_n$ ($a_n \to a$ and $b_n \to b$) we have $a_n \le b_n$ so $a \le b$.
but also $b_n \le a_n$ so $b \le a$ Thus $a = b$ $\square$.

If x_n is a sequence in $[a,b] = \{x \in \mathbb{R} : a \le x \le b\}$ and $x_n \to x$ as $n \to \infty$ then $x \in [a,b]$ $a \le x_n \Rightarrow \lim_{n\to\infty} a \le \lim_{n \to \infty} x_n \Rightarrow a \le x$
Likewise $x_n \le b \Rightarrow x \le b$.

Theorem: If a_n $(n \in I)$ converges to a limit then a_n is bounded.



Proof: let $\varepsilon = 1$ since $a_n \to L$ we know $\exists\, n_0 \in I$ s.t. $\forall n \in I$ $n > n_0$ $\Rightarrow |a_n - L| < 1 \Rightarrow L-1 < a_n < L+1$.

Consider the set $\{a_n : n \in I : n < n_0\} \cup \{L+1\}$. This set is finite, so it has a maximum R. for any n, either $n \leq n_0$ so $a_n \leq R$ because R is max

or $n > n_0$, so $a_n < L+1 \leq R$

Consider $\{a_n : n \in I\ n \leq n_0\} \cup \{L-1\}$. This is finite, so it has a minimum S. for any $n \in I$ either $n \leq n_0$ so that $a_n \geq S$

or $n > n_0$ so that $a_n > L-1 \geq S$

Thus $\forall n \in I$ $S \leq a_n \leq R$ $\square$.