Tuesday 14TH October 2008 MATH2039 - Analysis 3

Sequences

Definition: $a_n : n \in I$ is a Cauchy sequence if (*) $\forall \varepsilon > 0 \ \exists n_0 \in I$

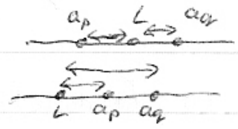
$\forall p, q \in I \quad p, q > n_0 \Rightarrow |a_p - a_q| < \varepsilon \qquad (|a_p - a_q| = |a_q - a_p|)$.

Theorem (Cauchy Criterion): If a_n converges to a limit L then a_n is Cauchy

Proof: We must show (*) Idea: If a_n tends to L then a_p, a_q will both be close to L if p, q are large so a_p, a_q will be close to each other.

write $|a_p - a_q| = |(a_p - L) + (L - a_q)| \leq |a_p - L| + |L - a_q|$

(triangle inequality) $\qquad (|L - a_q| = |a_q - L|)$.



Given $\varepsilon > 0$, as we know $a_n \to L$, there exists $n_0 \in I$ s.t. if $n > n_0$ then $|a_n - L| < \frac{\varepsilon}{2}$. Now if $p, q > n_0$ then $|a_p - L| < \frac{\varepsilon}{2}$, $|a_q - L| < \frac{\varepsilon}{2}$. so for $p, q > n_0$ we have

$|a_p - a_q| \leq |a_p - L| + |a_q - L| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$ $\square$.

This is called an $\frac{\varepsilon}{2}$ argument.

NOTE: If a_n is not a Cauchy sequence then a_n diverges (does not converge).

a_n is not a Cauchy sequence if $\exists \varepsilon > 0$ s.t. for all $n_0 \in I$ there exist $p, q > n_0$ with $|a_p - a_q| \geq \varepsilon$

Example: $a_n = n$, $n \in \mathbb{N}$ diverges

Proof: Cauchy Criterion fails for $\varepsilon = 1$. for any n_0 we can find ~~[illegible]~~ $p, q > n_0$ (e.g. $p = n_0 + 1$ $q = n_0 + 2$). Such that $|a_p - a_q| = |p - q| = 1 \geq \varepsilon$.

Example: $a_n = (-1)^n$ diverges.

Proof: a_n is not Cauchy for $\varepsilon = 2$. for any n_0 we can find $p, q > n_0$ Such that p is even and q is odd. Then $|a_p - a_q| = |(-1)^p - (-1)^q| = |1 - (-1)| = 2 \geq \varepsilon$.

Theorem: $\mathbb{R}$ satisfies Archimedean property $\forall x, y \in \mathbb{R}$ $x, y > 0$ $\exists n \in \mathbb{N}$ s.t. $n \cdot x > y$.

proof: Suppose for some $x, y > 0$ we have $n \cdot x \leq y$ for all $n \in \mathbb{N}$

Let $a_n = n \cdot x$ ~~[illegible]~~ This is bounded: $0 \leq n \cdot x \leq y$ for all n (by assumption). a_n is increasing because $a_{n+1} = a_n + x$ and $x > 0$.

Therefore a_n converges by the axiom.

On the other hand check Cauchy Criterion let $\varepsilon = x$. For any n_0 $\exists p, q > n_0$ (e.g. $p = n_0 + 1$ $q = n_0 + 2$) such that $|a_p - a_q| = |p \cdot x - q \cdot x| = x \geq \varepsilon$

Thus Cauchy Criterion fails so a_n diverges.

This is a contradiction, thus our assumption $x \cdot x \leq y$ is false so

$\exists n$ s.t. $n \cdot x > y$.

Definition: A sequence $a_n : n \in I$ diverges to ∞ if $\forall R > 0$ $\exists n_0 \in I$ s.t. $\forall n \in I$, $n > n_0 \Rightarrow a_n > R$.

A sequence a_n diverges to $-\infty$ if $\forall R < 0$ $\exists n_0 \in I$ s.t. $\forall n \in I$ $n > n_0 \Rightarrow a_n < R$

Example: $a_n = n$, $n \in \mathbb{N}$ diverges to ∞.

We must show $\forall R > 0$ $\exists n_0 \in \mathbb{N}$ s.t. $\forall n \in \mathbb{N}$, $n > n_0 \Rightarrow a_n > R$

$a_n > R$ means $n > R$,

By the Archimedean principle for $1, R$ $(x = 1, y = R)$

$\exists n_0 \in \mathbb{N}$ s.t. $n_0 \cdot 1 > R$. Now for any $n > n_0$ we have

$n > n_0 = n_0 \cdot 1 > R$ $\square$.

Alternatively: given $R > 0$ take $n_0 =$ integer part of R. Then $n > n_0 \Rightarrow n > R$.