Monday 13[TH] October 2008 MATH2039 - Analysis.

Sequences

Example: Let $c \in \mathbb{R}$. Let $a_n = c$ for $n = 1, 2, 3, \ldots$ then $a_n \to c$ as $n \to \infty$

Proof: check definition. We need to show ~~∀~~ $\forall \varepsilon > 0 \ \exists n_0 \in \mathbb{N}$ s.t. $\forall n \in \mathbb{N} \ n \geq n_0 \Rightarrow |a_n - c| < \varepsilon$ but $a_n = c$ so $a_n - c = 0$ given any $\varepsilon > 0$, choose $n_0 = 1$. Then $\forall n > n_0$ we have $|a_n - c| = |c - c| = 0 < \varepsilon$. The definition holds so $a_n \to c$ as $n \to \infty$ $\square$.

NOTE: We can take ANY n_0 we like.

Let $a_n : n \in I$ be a sequence let $S \subseteq I$. S infinite write $S = \{n_1, n_2, n_3, \ldots\}$. with $n_1 < n_2 < n_3 < \ldots$ ~~Example~~

Example: $I = \mathbb{N}$ $S \subseteq I$ is even numbers List S as $2, 4, 6, 8, \ldots$ $n_k = 2k$

we say that a_{n_k} is a subsequence of a_n

Example $a_n = \frac{1}{n}$ $n \in \mathbb{N}$ $S =$ even numbers then $a_{n_k} = a_{2k} = \frac{1}{2k}$.

$a_n : 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots$ $a_{2k} : \frac{1}{2}, \frac{1}{4}, \frac{1}{6}, \frac{1}{8}, \ldots$

Proposition If $a_n \to L$ as $n \to \infty$ and a_{n_k} is a subsequence then $a_{n_k} \to L$ as $k \to \infty$.

Proof: for simplicity let $I = \mathbb{N}$. To say $a_n \to L$ means $\forall \varepsilon > 0 \ \exists n_0 \in \mathbb{N}$. s.t. $\forall n \in \mathbb{N} \ n > n_0 \Rightarrow |a_n - L| < \varepsilon$. We have to show $a_{n_k} \to L$ that is $\forall \varepsilon > 0 \ \exists k_0 \in \mathbb{N}$ s.t. $k > k_0 \Rightarrow |a_{n_k} - L| < \varepsilon$.

key idea: n_k is always $\geq k$ $n_1 \geq 1$ and $n_{k+1} > n_k$

So $n_{k+1} \geq n_k + 1$ so $n_k \geq k \Rightarrow n_{k+1} \geq k + 1$.

Given $\varepsilon > 0 \ \exists n_0$ s.t. $n > n_0 \Rightarrow |a_n - L| < \varepsilon$. Let $k_0 = n_0$

If $k > k_0$ then $n_k > k_0 = n_0$ thus $|a_n - L| < \varepsilon$ holds for $n = n_k$

In other words $|a_{n_k} - L| < \varepsilon$ if $k > k_0$ $\square$.

We will show that if $a_n \to a$ and $b_n \to b$ then $a_n b_n \to ab$.

Assume this for now (check it later)

Example. $a_n = \frac{1}{n} \to 0$ as $n \to \infty$.

proof: a_n is bounded and monotonic so it converges

$\exists a$ s.t. $a_n \to a$ as $n \to \infty \Rightarrow$ subsequence $a_{2k} \to a$ as $k \to \infty$

but $a_{2k} = \frac{1}{2k} = \frac{1}{2}\left(\frac{1}{k}\right)$ $\frac{1}{k} \to a$ as $k \to \infty$ and $\frac{1}{2} \to \frac{1}{2}$ as $k \to \infty$.

$a_{2k} = a_k b_k$ $a_k = \frac{1}{k}$ $b_k = \frac{1}{2}$ $a_k \to a$ and $b_k \to \frac{1}{2}$ so $a_{2k} \to a \cdot \frac{1}{2} = \frac{1}{2} a$.

On the other hand we already said $a_{2k} \to a \Rightarrow a = \frac{1}{2} a$ so $2a = a$

so $a = 0$ so $\frac{1}{n} \to 0$ as $n \to \infty$ $\square$.

Example $a_n = \frac{1}{2^n} \to 0$ as $n \to \infty$

proof: a_n converges (bdd and monotonic). Look at subsequence

$a_{k+1} = \frac{1}{2^{k+1}} = \frac{1}{2}\frac{1}{2^k}$

$\lim_{k\to\infty} a_{k+1} = \lim_{n\to\infty} a_n$ $\qquad \lim_{k\to\infty} a_{k+1} = \frac{1}{2}\lim_{k\to\infty}\frac{1}{2^k} = \frac{1}{2}\lim_{n\to\infty} a_n$ $\qquad \lim_{n\to\infty} a_n = 0$ $\square$

Theorem: $\mathbb{R}$ has the Archimedean property. In other words $\forall x, y \in \mathbb{R}$

$x, y > 0$ $\exists n \in \mathbb{N}$ s.t. $n \cdot x > y$. we'll prove this later.

Example $a_n = \frac{1}{n} \to 0$ as $n \to \infty$

Second proof: We must show $\forall \varepsilon > 0$ $\exists n_0 \in \mathbb{N}$ s.t. $\forall n \in \mathbb{N}$

$n > n_0 \Rightarrow |a_n - 0| < \varepsilon$ but $|a_n - 0| = |\frac{1}{n} - 0| = \frac{1}{n}$

$\frac{1}{\varepsilon} < n \cdot 1 \Rightarrow \frac{1}{n} < \varepsilon$

By the Archamedean property for $x = 1$, $y = \frac{1}{\varepsilon}$ there exists a natural number n_0 such that $n_0 \cdot x > y$, i.e. $n_0 \cdot 1 > \frac{1}{\varepsilon}$

for any $n > n_0$ then $n > \frac{1}{\varepsilon}$ so $\frac{1}{n} < \varepsilon$.

We have shown that given any $\varepsilon > 0$ there is an n_0 s.t. $n > n_0 \Rightarrow \frac{1}{n} < \varepsilon$

i.e. $|a_n - 0| < \varepsilon$ Thus $a_n \to 0$ as $n \to \infty$ $\square$.