Thursday 9th October 2009 MATH 2039 - Analysis

Ordered fields

Definition: Let $A \subseteq F$ then A has maximum s if $s \in A$ and $\forall x \in A$ $x \leq s$.

A has minimum t if $t \in A$ and $\forall x \in A$ $x \geq t$.

Proposition: If $A \subseteq F$ is a FINITE SUBSET then A has a min and a max.

proof: Let $n = |A|$ = number of elements of A

If $n=1$ then $A = \{a\}$ let $s=a$ then $s \in A$. for any $x \in A$ we have $x = a = s$ so $x \leq s$. Similarly if $n=1$ then $t=a$ is minimum.

Induction step: Suppose $|A| = n+1$ and we know every set of size n has a maximum. Pick $a \in A$ and let $B = \{x \in A : x \neq a\}$ then $|B| = |A|-1 = n$ so B has a maximum. In other words $\exists s \in B$ s.t. $\forall x \in B$, $x \leq s$.

If $a > s$ then for every $x \in A$, either $x \in B$ so $x \leq s < a$ or $x = a$ so $x = a$.

Thus $\forall x \in A$ $x \leq a$ Also $a \in A$ so a is the maximum

Otherwise $a \leq s$. In this case s is the maximum because $s \in B \subseteq A$ so $s \in A$. Also $\forall x \in A$ either $x \in B$, so $x \leq s$ or $x = a$, so $x \leq s$. Thus we have shown if A has size $n+1$ then A has a maximum. By induction, for every n, a set of size n has a maximum, similarly for a minimum. $\square$

Definition F is Archimedean if $\forall x, y \in F$ with $x > 0$ and $y > 0$ then $\exists n \in \mathbb{N}$ s.t. $nx > y$. (y is not infinitely large, x is not infinitely small)

Proposition: $\mathbb{Q}$ is Archimedean

Proof: Let $x, y \in \mathbb{Q}$, $x, y > 0$ These are rational, and can be put over a common denominator. Write $x = \frac{a}{d}$, $y = \frac{b}{d}$, $a, b, d \in \mathbb{N}$

since $A > 0$ $nx > y \Leftrightarrow na > b$

$a \in \mathbb{N}$, so $a \geq 1$, so $na \geq n$ let $n = b+1$ then $na \geq b+1 > b$ $\square$.

Sequences and Limits

Definition: Let $I = \{m, m+1, m+2, \dots\}$ for some $m \in \mathbb{Z}$

(typically $m = 0, 1$ $I = \{0, 1, 2, 3, \dots\}$, $I = \{1, 2, 3, 4, \dots\}$).

A sequence is a function from I to F. We write this as $a_n : n \in I$

e.g. $a_0, a_1, a_2, \dots$ or $a_1, a_2, a_3, \dots$

Definition: $a_n : n \in I$ converges to a limit L if $\forall \varepsilon \in F$, $\varepsilon > 0$ ($\forall \varepsilon > 0$) $\exists n_0 \in I$ such that $\forall n \in I$ $n > n_0 \Rightarrow |a_n - L| < \varepsilon$.

Definition: If a_n does not converge to any limit L then a_n diverges.

Definition: a_n is bounded above if $\exists R \in F$ such that $a_n \leq R$ for all n.

a_n is bounded below if $\exists S \in F$ such that $S \leq a_n$ for all n.

a_n is bounded if $\exists R, S \in F$ such that $S \leq a_n \leq R$ for all n.

e.g. $a_n = n^2$ is not bounded above, $a_n = -n^2$ is not bounded below

$a_n = (-1)^n$ is bounded.

Definition: a_n is increasing if $\forall n \in I$ $\quad a_n < a_{n+1}$

non-decreasing $\quad a_n \leq a_{n+1}$

decreasing $\quad a_n > a_{n+1}$

non-increasing $\quad a_n \geq a_{n+1}$

a_n is monotonic if it is any of these four.

e.g. $a_n = n^2$ increasing $a_n = -n^2$ decreasing - both monotonic

$a_n = (-1)^n$ not monotonic.

Definition: The real numbers $\mathbb{R}$ are an ordered field satisfying the following axiom.

Axiom: Every bounded monotonic sequence in $\mathbb{R}$ converges to a limit in $\mathbb{R}$.

Example: the sequence $a_n = \frac{1}{n}$ converges

Why? It is decreasing (so monotonic) and bounded $0 < \frac{1}{n} \leq 1$ for all n.