Tuesday 7TH October 2008 MATH2039 - Analysis

Ordered Fields

Definition: A field is a set F with two operations $+, \cdot$ such that.

1) F with operation $+$ is an abelian group:
   - $\exists$ identity 0 s.t. $0+x=x+0=x$ for $x \in F$.
   - $\exists$ inverses so for each $x \in F$ $\exists (-x) \in F$ s.t. $x+(-x)=0$ $(-x)+x=0$
   - $+$ is associative $\forall x,y,z \in F$ $(x+y)+z=x+(y+z)$
   - $\forall x,y \in F$ $x+y=y+x$.
2) $\{x \in F : x \neq 0\}$ is an abelian group with operations
   - $\exists$ identity 1 s.t. $1 \cdot x = x \cdot 1 = x$ $\forall x \in F$
   - $\exists$ inverses x^{-1} for each $x \in F$ s.t $xx^{-1}=x^{-1}x=1$
   - $\cdot$ is associative $\forall x, y, z$ $(x \cdot y)\cdot z = x \cdot (y \cdot z)$
   - $\cdot$ is commutative $x \cdot y = y \cdot x$ $\forall x,y \in F$.
3) distributive law $\forall x,y,z \in F$ $x \cdot (y+z) = x \cdot y + x \cdot z$.

Examples $\mathbb{Q}$ (rational numbers), $\mathbb{R}$ (real numbers), and $\mathbb{C}$ (complex numbers) are all fields.

$\mathbb{N}$ (natural numbers) and $\mathbb{Z}$ (integers) are NOT fields.

Example $\forall x \in F$ a field $x \cdot 0 = 0$.

proof look at $x \cdot 0$ write $0=0+0$ (identity for $+$)

$x \cdot 0 = x \cdot (0+0) = x \cdot 0 + x \cdot 0$ (distributive law).

$x \cdot 0 + (-(x \cdot 0)) = (x \cdot 0 + x \cdot 0) + (-(x \cdot 0)) = x \cdot 0 + (x \cdot 0 + (-x \cdot 0))$ (associativity)

$0 = x \cdot 0 + 0$ (inverse) $0 = x \cdot 0$ (identity).

Definition An order on a set X is a relation $<$ such that.

1) trichotomy: $\forall x,y \in X$ exactly one of three things occurs:
   i) $x<y$ ii) $x=y$ iii) $y<x$.
2) transitive: If $x,y,z \in X$ and $x<y$ and $y<z$ then $x<z$.

Definition An ordered field is a field F with an order $<$ such that

- $\forall x,y,z \in F$ if $x<y$ then $x+z<y+z$
- $\forall x,y,z \in F$ if $x<y$ and $z>0$ then $x \cdot z < y \cdot z$

Definition for $x \in F$ ordered field. let $|x| = \begin{cases} x & \text{if } 0<x \\ 0 & \text{if } 0=x \\ -x & \text{if } 0>x \ (x<0). \end{cases}$

Notes: $|x| \geqslant 0$ for all x $(x<0 \Rightarrow x+(-x)<0+(-x) \Rightarrow 0<-x)$.

for all $x \in F$ $-|x| \leq x \leq |x|$

Proposition: 1) $\forall x, y \in F$ $|xy| = |x||y|$ 2) $\forall x, y \in F$ $|x+y| \leq |x|+|y|$ (triangle inequality)
3) $\forall x, y \in F$ $\big||x|-|y|\big| \leq |x-y|$. (reverse triangle inequality).

Proof 1). If $x=0$ or $y=0$ then $|x||y| = |xy| = 0$.

Otherwise we have four cases

If $x > 0$ and $y > 0$ then $x \cdot y > 0$ so $|x||y| = xy = |xy|$.

If $x < 0$ and $y < 0$ then $|x| = -x$ $|y| = -y$ these are positive so
$|x||y| = (-x)(-y) = xy$ is positive so $|xy| = xy = |x||y|$.

If $x < 0$ and $y > 0$ then $|x| = -x$ $|y| = y$ $|x||y| = (-x)y = -xy$
$|x|, |y|$ are positive so $-xy$ is positive so $xy < 0$ and $|xy| = -xy = |x||y|$

Last case $x > 0$ and $y < 0$ is similar.

2) We know $x \leq |x|$ and $y \leq |y|$ so $x+y \leq |x|+|y|$.
Also $x \geq -|x|$ so $-x \leq |x|$ and likewise $-y \leq |y|$ so
$-(x+y) = (-x)+(-y) \leq |x|+|y|$.
Since $|x+y|$ is either $x+y$ or $-(x+y)$ or 0 we get $|x+y| \leq |x|+|y|$

3) Write $x = y + (x-y)$ then $|x| = |y+(x-y)| \leq |y| + |x-y|$ by 2).
so $|x| - |y| \leq |x-y|$
Now we also have $|y| = |x+(y-x)| \leq |x| + |y-x| = |x| + |x-y|$.
so $|y| - |x| \leq |x-y|$
Finally $\big||x|-|y|\big|$ is one of $|x|-|y|$, $|y|-|x|$, 0
each of which is $\leq |x-y|$.