Monday 6TH October 2008 MATH2039 - Analysis

Introduction

Limits, convergence Sequences - list of numbers eg. 1, 2, 3, 4, 5, ...
eg. $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \ldots$ ②

② - goes to 0, how do we justify this? What does this actually mean?

Examples $a_n = \frac{n+1}{n}$ (index) $\frac{2}{1}, \frac{3}{2}, \frac{4}{3}, \frac{5}{4}, \ldots$ $b_n = \frac{n-1}{n}$ $\frac{0}{1}, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \ldots$

Thoughts on convergence:

1) 'More and more decimal places agree with the limit'.
2) 'Look at $|a_n - 1|, |b_n - 1|$ - 'More and more dp of $|a_n - 1|, |b_n - 1|$ are zero'.

for these two examples $(|a_n - 1|, |b_n - 1|)$ we can say that after 10+1 steps 1 dp of $|a_n - 1|, |b_n - 1|$ is 0 after 100+1 steps 2dp of $|a_n - 1|, |b_n - 1|$ are 0 ... after $10^n + 1$ steps n dp of $|a_n + 1|, |b_n - 1|$ are 0.

To say a number x is 0 to n dp means that $|x| < 10^{-n}$ or equivalently $-10^{-n} < x < 10^{-n}$

Let ε be a positive number (small), for a sequence a_n to tend to a limit L means $|a_n - L| < \varepsilon$ or equivalently $-\varepsilon < a_n - L < \varepsilon$. So long as n is large enough, for every $\varepsilon > 0$.

Definition: a_n <u>converges</u> to a limit L as $n \to \infty$ if for every $\varepsilon > 0$, there is some n_0 such that for all $n > n_0$ $|a_n - L| < \varepsilon$.

in symbols: $\forall \varepsilon > 0 \ \exists n_0 \in \mathbb{N}$ such that $\forall n > n_0$, $|a_n - L| < \varepsilon$.
$\forall$ = for all = for every $\exists$ = there exists

Lots of ways to make sequences

- Simplest way is a formula a_n = some function of n.
- Iteration Define x_0 ~~=~~ some value
  Define x_1 = function of x_0 x_2 = function of x_1

e.g. Let $x_0 = 2$ $x_1 = \frac{1}{2}(x_0 + \frac{2}{x_0})$ $x_2 = \frac{1}{2}(x_1 + \frac{2}{x_1})$ etc.
$x_n = \frac{1}{2}(x_{n-1} + \frac{2}{x_{n-1}})$

converges to $\sqrt{2}$ after 5 iterations.

for $n=5$ or greater $x_n = L$ (i.e. holds for 20 dp)
(for $n=5$ or greater $|x_n - L| < 10^{-20}$).

going from x_5 to x_6, value does not change (much).

$x_6 = L$, $x_5 = L$ $\quad x_6 = \frac{1}{2}(x_5 + \frac{2}{x_5})$ so $L = \frac{1}{2}(L + \frac{2}{L})$

$\Rightarrow 2L = L + \frac{2}{L} \quad L = \frac{2}{L} \quad L^2 = 2.$