Isomorphisms

$G \cong H$ if $\exists\ \varphi : G \to H$ bijection s.t. $\varphi(xy) = \varphi(x)\varphi(y)\ \forall x, y \in G$.

$(G, *), (H, \circ)$ $\varphi(x * y) = \varphi(x) \circ \varphi(y)$.

Examples 5.2. a) $\langle s \rangle \leq D_3$ $\langle s \rangle = \{e, s\}$ $\langle s \rangle \cong \mathbb{Z}_2 = \{[0], [1]\}$.

$\varphi : \langle s \rangle \mapsto \mathbb{Z}_2$ $e \mapsto [0]$ $s \mapsto [1]$ bij.

$\left.\begin{array}{l} \varphi(e \circ s) = \varphi(s) = [1] = [0] + [1] = \varphi(e) + \varphi(s) \\ \varphi(s \circ s) = \varphi(e) = [0] = [1] + [1] = \varphi(s) + \varphi(s). \end{array}\right\} \Rightarrow \varphi$ isomorphism.

$\Upsilon : \langle s \rangle \to \mathbb{Z}_2$ $e \mapsto [1]$ $s \mapsto e$ bijection but not isomorphism.

b) $\langle r \rangle \leq D_3$ $\langle r \rangle = \{e, r, r^2\}$ check $\varphi(r^i r^j) = \varphi(r^i) + \varphi(r^j)$.

$\varphi : \langle r \rangle \to \mathbb{Z}_3$ isomorphism $e \mapsto [0]$ $r \mapsto [1]$ hence $r^2 \mapsto [2]$

c) $D_3 \cong S_3$ $\varphi : D_3 \to S_3$ $r \mapsto (123)$ $s \mapsto (23)$

Defined on generators: make it into isomorphism:

$\varphi(rs) = (123)(23) = (12) = \varphi(r)\varphi(s)$ etc...

$\varphi(r^2) = (123)^2 = (132)$ $\varphi(e) = \varphi(s^2) = id = (23)(23) = \varphi(s)\varphi(s)$.

Proposition 5.3: Let G, H be groups and let $\varphi : G \to H$ be an isomorphism (i.e. $G \cong H$). a) $\varphi(e_G) = e_H$ b) Let $|g| = k$. Then $|\varphi(g)| = k$.

Proof a) $g \in G$ (G2) $e_G g = g$, $e_H h = h$ $\forall h \in H$.

$e_H \varphi(g) \underset{H}{=} \varphi(g) = \varphi(e_G g) \underset{\varphi \text{ isomorphism}}{=} \varphi(e_G)\varphi(g) \Rightarrow e_H \varphi(g) = \varphi(e_G)\varphi(g)$.

Use cancellation law in $H \Rightarrow e_H = \varphi(e_G)$.

b). Let $|g| = k$ and $|\varphi(g)| = l$

$e_H = \varphi(e_G) = \varphi(g^k) = \varphi(g^{k-1})\varphi(g) = \ldots = \varphi(g)^k$. $\Rightarrow l \mid k$.

φ bijection $\Rightarrow$ φ^{-1} exists and $\varphi^{-1}(\varphi(g)) = g$.

$e_G = \varphi^{-1}(e_H) = \varphi^{-1}(\varphi(g)^l) = [\varphi^{-1}(\varphi(g))]^l = g^l \Rightarrow k \mid l$

$\Rightarrow l = k$ and $|g| = |\varphi(g)|$ □. (Extra exercise: Show φ^{-1} is an isomorphism)

Example: $D_3 \cong S_3 \Rightarrow |r| = |(123)|$ $|s| = |(23)|$. But $D_3 \ncong \mathbb{Z}_6$ as $|[1]_6| = 6$. and D_3 has no element of order 6.

$|D_6| = 12$ orders 1, 2, 3, 6 & $|A_4| = 12$ orders are 1, 3, 2.

$|\mathbb{Z}_{12}| = 12$ cyclic $\Rightarrow |[1]| = 12$.

$D_6 \ncong A_4$ $A_4 \ncong \mathbb{Z}_{12}$ $\mathbb{Z}_{12} \ncong D_6$.

Theorem 5.4 (Classification of cyclic groups):

a) A cyclic group of order n is isomorphic to $\mathbb{Z}_n$.

b) An infinite cyclic group is isomorphic to $\mathbb{Z}$.

Proof: $G = \langle x \rangle = \{x^k \mid k \in \mathbb{Z}\}$ cyclic group

b). $\varphi: G \to \mathbb{Z}$ $x^k \mapsto k$ This is a bijection.

$\varphi(x^k x^l) = \varphi(x^{k+l}) = k+l = \varphi(x^k) + \varphi(x^l)$.

a) $|G| = n \Rightarrow G = \{e, x, x^2, \dots, x^{n-1}\}$.

$\varphi: \mathbb{Z}_n \to G$ $[i]_n \mapsto x^i$ bijection

but need to check φ is a well defined map.

$k \in [i]_n \Rightarrow k = nq + i$ some $q \in \mathbb{Z}$.

$\varphi([k]) = x^k = x^{nq+i} \underset{\substack{\uparrow \\ \text{law of exponents in } G}}{=} (x^n)^q \cdot x^i = \underset{\substack{\uparrow \\ x^n = e_G}}{e_G} \, x^i = x^i = \varphi([i]_n)$.

$\Rightarrow \varphi$ is well defined.

$\varphi([i]_n + [j]_n) = \varphi([i+j]_n) = x^{i+j} \underset{\substack{\uparrow \\ \text{law of exponents}}}{=} x^i x^j = \varphi([i]_n)\, \varphi([j]_n)$.

$\Rightarrow \varphi$ is an isomorphism $\square$.