Tuesday 21st October 2008 MATH 2003 - Group Theory

Permutations

$\sigma = \sigma_1 \ldots \sigma_m$ disjoint cycle decomposition of σ in S_n

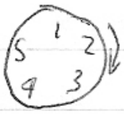
Definition 3.15 $\text{sign}(\sigma) = (-1)^{n-m}$ (careful not to forget 1-cycles)

3.16 (b) l_i = length σ_i $\quad \text{sign}(\sigma) = \prod_{i=1}^{m} (-1)^{l_i - 1}$

Theorem 3.17 $\sigma = \tau_1 \ldots \tau_k$ Product of transpositions $\Rightarrow \text{sign}\,\sigma = (-1)^k$

k even $\Leftrightarrow \text{sign}(\sigma) = 1$ $\quad$ k odd $\Leftrightarrow \text{sign}(\sigma) = -1$

τ transposition $\tau = (ab) \overset{3.16(b)}{\Longrightarrow} \text{sign}(\tau) = -1$



$(12345) = (23451)$

$= (34512) = \ldots$

recall $(a_1 a_2 \ldots a_k) = (a_2 a_3 \ldots a_k a_1) = \ldots$

Proof of 3.17: Induction on k: $k=1 \Rightarrow \text{sign}(\sigma) = \text{sign}(\tau_1) = -1$

$k > 1$ Suppose $\text{sign}(\tau_2 \tau_3 \ldots \tau_k) = (-1)^{k-1}$

Need to show $\text{sign}(\tau_1 \tau_2 \ldots \tau_k) = (-1)\,\text{sign}(\tau_2 \tau_3 \ldots \tau_k)$

Show something slightly more general $\forall \sigma \in S_n$, τ transposition

$\text{sign}(\tau\sigma) = -\text{sign}(\sigma)$ in other words show that the difference between the number of cycles in the disjoint cycle decompositions of σ and $\tau\sigma$ is exactly 1.

Def $\sigma = \sigma_1 \ldots \sigma_m$ $\quad \sigma \in S_n$ $\quad \text{sign}(\sigma) = (-1)^{n-m}$

Let $\tau = (ab)$, $\sigma = \sigma_1 \ldots \sigma_m$. Look at those σ_i s.t.

$a, b \notin \sigma_i = (s_1 \ldots s_{l_i}) \Rightarrow \sigma_i$ is also a cycle in the disjoint cycle notation of $\tau\sigma$.

Case 1 $a, b \notin \sigma_i$ $\forall i = 1, \ldots, m$ $\quad \tau\sigma = \tau\sigma_1 \ldots \sigma_m$ disjoint cycle decomposition.

Case 2: $a \in \sigma_i$, $b \in \sigma_j$ $(i \neq j)$ disjoint cycles commute, so we may assume

$a \in \sigma_1$ $b \in \sigma_2$. $\sigma_1 = (a_0 a_1 \ldots a_s)$ $a = a_i$ some i σ cycle $\Rightarrow$ wlog $a = a_0$

wlog $\sigma_2 = (b_0 b_1 \ldots b_t)$ and $b = b_0$ $\Rightarrow \sigma = (a a_1 a_2 \ldots a_s)(b b_1 \ldots b_t)\sigma_3 \ldots \sigma_k$

$\tau\sigma = (ab)(a a_1 \ldots a_s)(b b_1 \ldots b_t)\sigma_3 \ldots \sigma_k = (b b_1 \ldots b_t a a_1 \ldots a_s)\sigma_3 \ldots \sigma_k$

1 less cycle than σ.

Case 3: $a, b \in \sigma_i$ wlog $\sigma_i = \sigma_1$ wlog $\sigma_1 = (a a_1 \ldots a_s b b_1 \ldots b_t)$

$\tau\sigma = (ab)(a a_1 \ldots a_s b b_1 \ldots b_t)\sigma_2 \ldots \sigma_k = (a a_1 \ldots a_s)(b b_1 \ldots b_t)\sigma_2 \ldots \sigma_k$

1 more cycle than σ $\square$

Examples $\pi = (123) = (13)(12) \Rightarrow \text{sign}(\pi) = (-1)^2 = 1$

$id = (12)(12)$ even $\Rightarrow \text{sign}(id) = 1$

$\pi = (123)(789) = (13)(12)(79)(78)$ $\quad \text{sign}(\pi) = (-1)^4 = 1$.

Lemma 3.19 $\sigma, \pi \in S_n$ $\quad$ i) $\text{sign}(\sigma\pi) = \text{sign}(\sigma)\,\text{sign}(\pi)$

ii) $\text{sign}(\sigma^{-1}) = \text{sign}(\sigma)$

Proof: Exercise.

Definition 3.20 We denote by A_n the set of all even permutation is S_n

Proposition 3.21 A_n is a group, under composition (of maps), called the Alternating Group of degree n.

Proof: $\sigma, \pi \in A_n \Rightarrow \text{sign}(\sigma) = \text{sign}(\pi) = 1$ lemma 3.19 $\Rightarrow \text{sign}(\sigma \circ \pi) = \text{sign}(\sigma)\text{sign}(\pi)$
$= 1 \cdot 1 = 1$

$\Rightarrow \sigma \circ \pi \in A_n \Rightarrow \circ$ is a binary operation.

(G1) Associativity follows from G1 for S_n

(G2) $id_{A_n} = id_{S_n}$ as $id_{S_n} \in A_n$ $\text{sign}(id) = 1$

(G3) $\sigma \in A_n \Rightarrow \sigma^{-1} \in A_n$ lemma 3.19 $\text{sign}(\sigma^{-1}) = \text{sign}(\sigma) = 1$.

Nothing comparable for odd permutations $(12)(34)$ is even but (12) and (34) are odd

Example $A_2 = \{id\} \subseteq S_2 = \{id, (12)\}$.

$A_3 = \{id, (123), (132)\} \subseteq S_3 = \{id, (123), (132), (12), (13), (23)\}$.

Theorem 3.22: A_n has $\frac{n!}{2}$ elements

Proof: We show that each $\pi \in S_n$ can be written either as $\pi \in A_n$ or as $\pi = \sigma(12)$ where $\sigma \in A_n$. Suppose this is true.

$S_n = A_n \cup A_n(12)$ disjoint union

$|S_n| = |A_n| + |A_n(12)|$ $A_n(12) = \{\sigma(12) \mid \sigma \in A_n\}$

$= 2|A_n| \Rightarrow |A_n| = \frac{|S_n|}{2} = \frac{n!}{2}$

Prove Claim. $\text{sign}\,\pi = \begin{cases} 1 \Rightarrow \pi \in A_n \\ -1 \overset{3.19}{\Rightarrow} \pi(12) \in A_n \end{cases}$ $\Rightarrow \pi = \underbrace{\pi(12)}_{\in A_n}\overbrace{(12)(12)}^{id}$

$\text{sign}(\pi(12)) = (-1)(-1)$

$= \sigma(12)$ with $\sigma = \pi(12) \in A_n$ $\square$

$\sigma \in S_n$ 'all powers of σ': $\sigma^0, \sigma^1, \sigma^2, \dots, \sigma^{m-1}$ where m order of σ.

Linear Algebra cycles $\langle i_1, i_2, \dots, i_r \rangle$ } same thing.

Group Theory cycles $(i_1\, i_2 \dots i_r)$