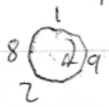
Thursday 16th October 2008   MATH2003 - Group Theory

Permutations

$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 9 & 8 & 3 & 6 & 5 & 7 & 4 & 1 & 2 \end{pmatrix} = (1928)(467) = (467)(1928).$

Theorem 3.10 Every permutation is the product of disjoint cycles and this decomposition is unique.   Proof in 1049.



$(1928)^{-1} = (8291)$

Definition 3.11: order of a permutation $\pi \in S_n$ is the smallest integer $m \geq 1$ s.t. $\pi^m = id$.

Lemma 3.12: The order of an r-cycle is r.

proof: $i_1, \ldots, i_r \in \{1, \ldots, n\}$ $\pi(i_j) = i_{j+1}$ $\pi(i_r) = i_1$ $j = 1, \ldots, r-1$.
and for all $i_j \in \{i_1, \ldots, i_r\}$ $\pi^r(i_j) = i_j$ $\forall s < r$ s.t. $\pi^s(i_j) \neq i_j$.

$\pi = (17824)(365) = \pi_1 \pi_2$ $[\pi^2 = (\pi_1\pi_2)^2 = \pi_1\pi_2\pi_1\pi_2 \underset{\pi_1,\pi_2 \text{ disjoint}}{=} \pi_1^2 \pi_2^2]$.

$\pi_1^5 = id$ $\pi_1^{10} = id$ π_1^{15} id $\pi_2^3 = id$ $\pi_2^6 = id$ $\pi_2^9 = id$ $\pi_2^{12} = id$ $\pi_2^{15} = id$.

$(1234)(56)$ $\pi_1^4 = id$ $\pi_2^2 = \pi_2^4 = id$.

Theorem 3.13 The order of a permutation $\pi \in S_n$ is the least common multiple of the lengths of the cycles in the disjoint cycle decomposition of π.

Proof: $\pi = \pi_1 \pi_2 \ldots \pi_k$ disjoint cycles $\xrightarrow[\text{commute (3.9)}]{\text{disjoint cycles}}$ (Let m be the smallest integer ≥ 1 s.t. $\pi^m = id$) $\pi^m = \pi_1^m \pi_2^m \ldots \pi_k^m$ and $\pi^m = id \Leftrightarrow \pi_i^m = id$ $\forall i = 1, \ldots, n$.
$\Leftrightarrow m = \text{lcm}$ (orders of the π_i) $\underset{3.12}{\Leftrightarrow}$ lcm (length of π_i) $\square$

$(12345) = (23)(34)(45)(15) = (15)(14)(13)(12) = (13)(23)(34)(45)(12)(15)$.

Theorem 3.14: Every ~~product~~ permutation is a product of transpositions.

Proof: By 3.13 it's enough to prove claim for cycles

- $id = (12)(12)$
- $(i_1 i_2 \ldots i_{r-1} i_r) = (i_1 i_r)(i_1 i_{r-1}) \ldots (i_1 i_3)(i_1 i_2)$

Definition 3.15. Let $\sigma \in S_n$ and let $\sigma = \sigma_1 \ldots \sigma_m$ be the disjoint cycle decomposition of σ. Then $\text{sign}(\sigma) = (-1)^{n-m}$ is called the sign of σ. If $\text{sign}(\sigma) = 1$ we call σ an even permutation. If $\text{sign}(\sigma) = -1$ we call σ an odd permutation.

$\sigma = (123)(456) \Rightarrow \text{sign}(\sigma) = (-1)^{6-2} = (-1)^4 = 1$ $\sigma \in S_6$.

$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 2 & 3 & 1 & 5 & 6 & 4 & 7 \end{pmatrix} \in ?$

Remark 3.16: a) Let σ be an r-cycle. Then $\text{sign}(\sigma) = (-1)^{r-1}$ (from 1049).
b) $\sigma = \sigma_1 \ldots \sigma_m$ disjoint cycle decomposition. Then $\text{sign}\,\sigma = \prod_{i=1}^{m} (-1)^{l_i - 1}$ where l_i = length of σ_i.

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 2 & 3 & 1 & 5 & 6 & 4 & 7 \end{pmatrix} \in S_7$$

$= (123)(456)(7)$ $\quad$ $\text{sign}(\sigma) = (-1)^{7-3} = (-1)^4 = 1$

Definition 3.15 should be used with care (don't forget 1-cycles).

or, alternatively, use 3.16 b).

or, use:

Theorem 3.17: Let $\tau_1, \dots, \tau_r$ be transpositions in S_n. Then $\text{sign}(\tau_1 \dots \tau_r) = (-1)^r$

Remark: This is often used as definition of sign but you have to prove this is independent of product chosen $\pi = \tau_1 \dots \tau_r = \nu_1 \dots \nu_s$

$\text{sign}(\pi) = ?$ $\quad$ $\text{sign}(\text{transposition}) = -1$

Corollary 3.18 Let $\sigma \in S_n$. Let $\sigma = \tau_1 \dots \tau_r$ be an arbitrary decomposition into a product of transpositions. Then $\text{sign}(\sigma) = 1 \Leftrightarrow r$ even or $\text{sign}(\sigma) = -1 \Leftrightarrow r$ odd.

In particular, for every permutation, any decomposition into a product of transpositions has either an even number of factors or an odd number of factors.