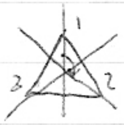
Wednesday 15TH October 2008 MATH2003 - Group Theory

Permutations.

$Sym(X) = S(X)$ $\quad S_n \quad \{x_1, \dots, x_2\} \xleftrightarrow[bij]{} \{1, 2, \dots, n\}$.

$X = \{1, \dots, n\}$ $\quad i \in X \quad \pi \in S_n \quad \pi(i) \in X \quad \pi = \begin{pmatrix} 1 & 2 & \cdots & n \\ \pi(1) & \pi(2) & \cdots & \pi(n) \end{pmatrix}$.



$\pi = r$: $1 \to 2 = \pi(1)$, $2 \to 3 = \pi(2)$, $3 \to 1 = \pi(3)$ $\quad \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix} = \pi$ moves all $i \in \{1, 2, 3\}$.

$\pi = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 2 & 1 & 4 & 5 \end{pmatrix}$ moves 1, 2, 3 fixes 4, 5.

Proof of Lemma 3.4:

$$\left.\begin{array}{ll} \pi(1) \in \{1, \dots, n\} = X & n \text{ choices} \\ \pi(2) \in X \setminus \{\pi(1)\} & n-1 \text{ choices} \\ \vdots & \\ \pi(n) \in X \setminus \{\pi(1), \pi(2), \dots, \pi(n-1)\} & 1 \text{ choice} \end{array}\right\} n(n-1)\dots 1 = n!$$

$S_3 \cong D_3$ has 6 elements $3! = 6$. S_4 has 24 elements ...

$D_4 \not\subseteq S_4$ $\quad |D_4| = 8 \quad |S_4| = 24$.

How to multiply permutations.

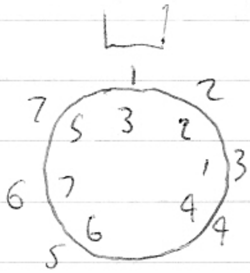
$\pi = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 1 & 2 & 4 \end{pmatrix}$ $\quad \sigma = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \end{pmatrix}$ $\quad \pi, \sigma$ are maps $\pi \circ \sigma$: σ first, then π

$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 1 & 2 & 4 \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 3 & 4 & 2 \end{pmatrix}$ $\quad 1 \xrightarrow{\sigma} 2 \xrightarrow{\pi} 1$, $\quad 2 \to 1 \to 3$

$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 1 & 2 & 4 \end{pmatrix}^{-1}$ is an element $\pi^{-1} \in S_4$ s.t. $\pi^{-1}\pi = id = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \end{pmatrix}$.

π: $1 \xrightarrow{\pi} 3 \xrightarrow{\pi^{-1}} 1$, $2 \to 1 \to 2$, $3 \to 2 \to 3$, $4 \to 4 \to 4$ $\quad \Rightarrow \pi^{-1} = \begin{pmatrix} 3 & 1 & 2 & 4 \\ 1 & 2 & 3 & 4 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 1 & 4 \end{pmatrix}$.

Proof of Lemma 3.5. $\pi^{-1}\pi(i) = i$.



$3 \to 2 \to 1 \to 4 \to 6 \to 7 \to 5 \to 3$

cycle: $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 2 & 5 & 4 & 6 & 1 \end{pmatrix}$

$\{1, 2, \cancel{6}, 6\}$ $\quad \{1, 3, 5, 6\}$.

$\pi(1) = 3 \quad \pi(3) = 5 \quad \pi(5) = 6 \quad \pi(6) = 1$

and fixes 2 and 4.

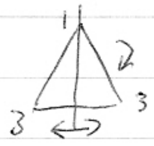
$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 4 & 5 & 2 & 6 & 1 \end{pmatrix}$ <u>not</u> a cycle $= \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 2 & 5 & 4 & 6 & 1 \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 4 & 3 & 2 & 5 & 6 \end{pmatrix}$

$\{2, 4\}$ $\quad 2 \to 4 \to 2$.

How to write cycles

$\begin{pmatrix}1&2&3&4&5&6\\3&2&5&4&6&1\end{pmatrix} = (1\,3\,5\,6)$ $\quad 1\to3\to5\to6\to1$

$r \in S_3 = D_3$



$r = \begin{pmatrix}1&2&3\\2&3&1\end{pmatrix} = (1\,2\,3)$ $\quad 1\to2\to3\to1$

$s = \begin{pmatrix}1&2&3\\1&3&2\end{pmatrix} = (1)(23) = (23)$ $\quad 2\to3\to2$

ignore single element cycles.

How to multiply cycles

1) transform into matrix notation - multiply - transform back (cumbersome.)

2) $(123)(23) = (21)(3) = (21)$ go right to left.

$2\to3\to1 \quad 1\to2 \quad 3\to2\to3$

$3\to2\to3 \quad 2\to3\to1 \quad 1\to2 \quad (3)(21) = (21) = (12)$

$2\to1\to2 \quad 1\to2\to1$

$(23)(123) = (13)(2) = (13)$

$sr \neq rs$

$(123)(45) = (45)(123)$ as disjoint cycles.

Proof of proposition 3.9.

$\pi, \sigma \in S_n$ disjoint $X = \{1, 2, \dots, n\}$.

$\Rightarrow A \subseteq X$ s.t. π moves all elements of A, σ fixes all elements of A.

$B \subseteq X$ s.t. σ moves all elements of B, π fixes all elements of B

$C \subseteq X$ s.t. π, σ fix all elements of C.

need to show $\forall i \in X$ $\pi\sigma(i) = \sigma\pi(i)$

i) $a \in A$ $\pi\sigma(a) = \pi(a) = \sigma\pi(a)$ as $\pi(a) \in A$ and σ fixes every element in A.

ii) $b \in B$ $\pi\sigma(b) = \sigma(b)$ as $\sigma(b) \in B$, π fixes B $= \sigma\pi(b)$ as $\pi(b) = b$.

iii) $c \in C$ $\pi\sigma(c) = c = \sigma\pi(c)$ $\square$