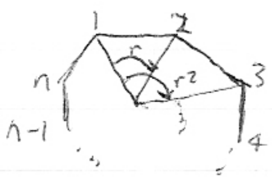
Semigroups and Monoids

Example 1.3 a) $x, y, z \in X$ $(x * y) * z = y * z = z$
$x * (y * z) = x * z = z$ $\Rightarrow$ semigroup.
But if $e * x = x$ then $x * e = e$ ↯ - no identity.
b) Multiplication associative $1 = e$ not a group as
$2^{-1} = \frac{1}{2} \notin \mathbb{Z}$ - (G3) fails.
c) $2\mathbb{Z} = \{2n \mid n \in \mathbb{Z}\}$ multiplication associative
but $1 \notin 2\mathbb{Z}$ so not monoid.

Equilateral triangle symmetry: $\{e, r, r^2, s, u, t\}$
$sr = u$ but $rs = t$ so not abelian
Symmetry groups of $\boxed{T}$ and $\triangle$ finite $|T| = 4$ $|D| = 6$
$\mathbb{Z}$ infinite.

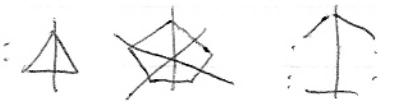
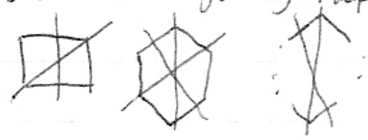
$f: \mathbb{R}^n \to \mathbb{R}^n$ a bijective function. f is called an ISOMETRY if it preserves distances.
$\emptyset \neq X \subseteq \mathbb{R}^n$ (usually $X \subseteq \mathbb{R}^2, \mathbb{R}^3$)
$f: X \to X$ is an isometry, we call it a symmetry
Theorem 1.6 $\emptyset \neq X \subseteq \mathbb{R}^n$, then the set of symmetries of X is a group under composition of maps (denoted by $\Sigma(X)$).
$\Sigma(X) \subseteq S(X)$ ~ set of all permutations of X.
If $|X| = n \Rightarrow \Sigma(X) \subseteq S_n$ $(Sym(X) = S(X))$
Proof: Composition is a binary operation: $X \xrightarrow{f} X \xrightarrow{g} X$ and $g \circ f$ is a bijection* as f and g are. *Actually a symmetry.
(G1) Composition of maps is associative (from Linear Algebra)
(G2) $e = id_X$ the identity map (this is a symmetry)
(G3) Bijections have inverses (see exercises) ~ these are symmetries.

Symmetries of regular n-gons.
label vertices - each symmetry permutes these $(\Sigma(X) \subseteq S_n)$.



• $(n-1)$ non trivial rotations about $\frac{2\pi}{n}$
$r, r^2, \ldots, r^{n-1}$

- identity
- n reflections case 1 n odd : n reflections in line joining vertex with midpoint of opposite edge
  case 2 n even : $\frac{n}{2}$ reflections in line joining opposite vertices
  $\frac{n}{2}$ reflections in line joining midpoints of opposite edges.

odd:  even: 

in total $2n$ symmetries - dihedral group : D_n $|D_n| = 2n$

D_n permutes n vertices $\Rightarrow D_n \subseteq S_n$
$n=3$ $\triangle$ $D_n \cong S_n$ ($\cong$: essentially the same).
$n>3$. $|D_n| = 2n$ $|S_n| = n!$ $2n < n!$

proposition 1.9 a) $e * g = g$ and $e' * g = g \Rightarrow e = e'$.
Corollary 1.11 $[(ab)c]^{-1} = c^{-1}(ab)^{-1} = c^{-1}b^{-1}a^{-1}$

Theorem 1.13 - proof : a) $ax = bx \Rightarrow a = b$
$ax = bx \overset{G3}{\Rightarrow} (ax)x^{-1} = (bx)x^{-1} \overset{G1}{\Rightarrow} a(xx^{-1}) = b(xx^{-1}) \overset{G3,G2}{\Rightarrow} ae = be \overset{G2}{\Rightarrow} b = a$.
b) $ax = b$ has UNIQUE solution
Existance $x = a^{-1}b$
Uniqueness : Suppose also $ay = b$ $\Rightarrow ay = ax$ now use part a)
and cancel $\Rightarrow y = x$.

Example 1.14. $(\mathbb{Q}^\times, *)$ $\frac{ab}{2} \in \mathbb{Q} \Rightarrow *$ binary
Associativity (G1): $(a*b)*c = (\frac{ab}{2})*c = \frac{abc}{4}$
$a*(b*c) = a*(\frac{bc}{2}) = \frac{abc}{4}$
Identity (G2) $e=2$: $a*2 = \frac{a \cdot 2}{2} = a$
Inverse (G3) $a^{-1} = \frac{4}{a}$ $a * \frac{4}{a} = \frac{a \cdot \frac{4}{a}}{2} = 2\ (=e)$.

$4 = 2 * x = \frac{2x}{2} = x$.